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M. A./M. Sc. Examination-2021
Semester-III
Mathematics
Optional Course : MMO-31 (P03)
(Advanced Real Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **question no.6** and any **three** from the rest

1. (a) Show that the set $\{x : \omega(x) < \epsilon\}$ is open for every $\epsilon > 0$. [3]
   (b) Show that for any function f the set of all points at which f assume a strict maximum or strict minimum is countable. [3]
   (c) Define Dini derivates of a function f at a point a. Show that the set $\{x : D^+f(x) < D_-f(x)\}$ is countable [2+4]
2. (a) Suppose f is continuous and $D^+f(x) \geq 0$ except on a countable set in $[a, b]$. Then show that f is non decreasing in $[a, b]$ [3]
   (b) If all Dini derivates of function f is finite at x then show that f is continuous at x. [4]
   (c) Define Cantor set on $[0, 1]$ and show that it is closed and has measure zero. [2+3]
3. (a) Show that if a function f is of bounded variation, continuous and satisfies Lusin condition on [a,b] then f is absolutely continuous on $[a, b]$. [9]
   (b) Show that an absolutely continuous function maps measurable sets into a measurable sets. [3]
4. (a) What is Vitali cover. State and prove Vitali's covering theorem [1+6]
   (b) Suppose f is monotone increasing in $[a, b]$ then show that f' is Lebesgue integrable on $[a, b]$ and $0 \leq \int_0^1 f'(x)dx \leq f(b) - f(a)$. [5]
5. (a) Show that an f is usc function at x if and only if $f(x) \geq \overline{f}(x)$ [4]
   (b) Define Baire one function. Show that if f has finite number of discontinuities in $[a, b]$ then f is a Baire one function. [5]
   (c) What is tagged partition, Define Hanstock integral of a function. [3]
6. Answer any two
   (a) If f is of bounded variation on $[a, b]$ then show that f' exists $a.e$ on $[a, b]$ [2]
   (b) Show that Cantor function is not absolutely continuous. [2]
   (c) Define convex function.Give example of a convex function which is not continuous on $[0, 1]$ [2]

M.A./M.Sc. Examination-2021

Semester-III

Mathematics

Optional Paper: MM0-31(P 08/A 07)

(Lie Theory of Ordinary and Partial Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **Question No. 1** and any **three** from the rest five.

1. Find infinitesimal generators for the Lie group of (point) symmetry transformations admitted by the partial differential equation $u_t(x,t) = c(u(x,t)u_x(x,t) + u_{xx}(x,t))$. Give the rule of transformations for each infinitesimal generator. [9+4]

   or,

   Derive the determining equations for coefficients involved in infinitesimal generators for the Lie group of (point) symmetry transformations admitted by the KdV (partial differential) equation in (1+1) dimensions. Show that determining equations have solutions admitting translations in space and time variables as Lie group of symmetry transformations of the above equation. Reduce the KdV equation to an ordinary differential equation (ODE) with appropriate similarity variables. [9+4]

2. Define Lie group and Lie group of transformations.Vefify whether the transformations $(x,y) \to (x^*,y^*) = \left(\frac{x}{1+\epsilon_1 x+\epsilon_2 y}, \frac{y}{1+\epsilon_1 x+\epsilon_2 y}\right)$ may be regarded as Lie group of transformations or not. Construct commutator table for the infinitesimal generators, if yes. [1+5+3]

3. Find the infinitesimal generators for the Lie group of point transformations $(x,y) \to (x^*,y^*) = (x\cos\epsilon_3 - y\sin\epsilon_3 + \epsilon_1,\ x\sin\epsilon_3 + y\cos\epsilon_3 + \epsilon_2)$. Find the first and second prolongation for each infinitesimal generators. [1+5+3]

4. Define differential invariant for a Lie group of point transformations. Find classes of first and second-order ODEs invariant under the Lie group of point transformations generated by $\hat{X} = y\,\partial_y$. [2+3+4]

5. Verify whether the ODE, $(x^2+y^2)y'' + 2(y - xy')(1+y'^2) = 0$ admits the rotation group as its Lie group of symmetry transformations. Reduce the order of the equation, if yes. [6+3]

6. What do you mean by invariant surface conditions for a boundary value problem? State conditions precisely. Hence find the group invariant solution to the heat equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ corresponding to the Lie group of transformations generated by the infinitesimal generator $\hat{X} = x\,\partial_x + 2t\,\partial_t - 2u\,\partial_u$. [1+2+6]

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Use separate answer script for each unit

M. Sc. Examination-2021

Semester-III
Mathematics
Elective Course: MMC-35
(Galois Theory -I and Multivariable Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Galois Theory I)

Answer *any two* questions.

1. (a) Define a simple extension of a field. Let F be a field and K be an extension of F. Then show that an element $\alpha \in K$ is algebraic over F if and only if the extension $F(\alpha)/F$ is finite. [5]
   (b) What is composite of two field extensions of a field? Show that the composite of the exetnsions $Q(\sqrt{5})$ and $Q(\sqrt{7})$ is a simple extension of Q. [5]
2. (a) What do you mean by a splitting field of a polynomial over a field? Explain by an example. Determine which of the fields $\mathbb{Z}_3(i)$ and $\mathbb{Z}_3[x]/ < f(x) >$ is/are splitting field(s) of the polynomial $f(x) = x^2 + x + 2$ over the field $\mathbb{Z}_3$. [2+4]
   (b) Let K be a field extension of a field F and let the elements $a, b \in K$ be algebraic over F of degree m and n (i.e. $\deg(\min(F, a)) = m$ and $\deg(\min(F, a)) = n$) respectively. Prove or disprove: $[F(a, b) : F] = mn$. [4]
3. (a) Find the splitting field of K of $x^8 - 3$ over Q. Determine the Galois group $Gal(K/Q)$ of the extension K/Q and hence examine whether K/Q is a Galois extension. [3+5]
   (b) Let F be a field and K be its extension of prime degree. If $a \in K$, then show that either $F(a) = F$ or $F(a) = K$. [2]

Unit-II (Full Marks: 20)
(Multivariable Analysis)

Answer **any two** questions

1. (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ be such that for some matrix T, $\lim_{n\to\infty} \frac{||f(x)-f(a)-T(x-a)||}{||x-a||} = 0$ then show that T is the matrix of partial derivatives at a. [4]
   (b) Suppose $f : A \to \mathbb{R}^m$ is differentiable at $x_0 \in A$ where $A \subset \mathbb{R}^n$ be open set. Show that there exists M and δ such that $||f(x) - f(x_0)|| \leq M||x - x_0||$ for every $x \in A$ satisfying $||x - x_0|| < \delta$. [4]
   (c) Compute $f'(0)$ for the function $f(x, y) = (\sin x, \cos y, \sin y, \cos x, e^{xy})$ [2]
2. (a) Suppose $f : A \to \mathbb{R}^m$ has third order continuous partial derivatives at $x_0 \in A$ where $A \subset \mathbb{R}^n$ be open set.Then show that

$$f(x) = f(x_0) + \sum(\frac{\partial f}{\partial x_i}(x_0))(x_i - X_i) + Hf(x_0)(x - x_0) + R_2(x) \quad \text{and} \quad \lim_{x\to x_0} \frac{|R_2(x)|}{||x - x_0||^2} = 0$$

   [6]
   (b) Define directional derivative $D_v f(a)$ of the function $f : \mathbb{R}^n \to \mathbb{R}^m$. Show that $\{D_v f(a) : v \in \mathbb{R}^n\}$ is a linear subspace of $\mathbb{R}^m$ [4]
3. (a) Suppose $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^n$ be open and $x_0 \in A$ is a local minimum point of f then show that the Hessian matrix is positive semi-definite. [4]
   (b) State and prove inverse function theorem for the function $f : \mathbb{R}^N \to \mathbb{R}^N$ [5]
   (c) Give example of an open map $f : \mathbb{R}^N \to \mathbb{R}^N$ with proper justification. [1]

S. Kalla

Use separate answer script for each unit

M.Sc. Examination - 2021

Semester-III
Mathematics
Course: MMC-32
(Advanced Mathematical Statistics - I and Theory of Chaos)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Notations and symbols have their usual meanings

Unit-I : Marks - 20
(Advanced Mathematical Statistics - I)

Answer ***any two*** questions

1. Describe the χ^2 one-sample hypothesis test. In a 8-horse race, test the effect of post positions by analyzing the 144 race results in 18 days of racing for the first month of 2021 as given in the following table, assume $\alpha = 0.01$, the significance level of the test.

Post positions	1	2	3	4	5	6	7	8	
No. of wins	29	19	18	25	17	10	15	11	Total 144

It is given that $\chi^2(7) > 14.067 = 0.05$ and $\chi^2(7) > 16.622 = 0.02$. [4+6]

2. (a) Show that in Normal regression analysis the likelihood estimators $\hat{\beta}$ and $\hat{\alpha}$ are unbiased estimators of β and α, the regression co-efficients. In this connection, estimate the parameter σ^2. [6]

(b) Show that $\frac{\hat{\beta}-\beta}{\hat{\sigma}}\sqrt{\frac{n-2}{n}S_{xx}}$ and $\frac{\hat{\alpha}-\alpha}{\hat{\sigma}}\sqrt{\frac{(n-2)S_{xx}}{n(\bar{x})^2+S_{xx}}}$ have both t-distribution with $(n-2)$ degrees of freedom. [4]

3. Consider the standard model of one-way ANOVA given by

$$Y_{ij} = \mu_i + \epsilon_{ij} \quad \text{for } i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

where $m \geq 2$ and $n \geq 2$. Assume that each random variable

$$Y_{ij} \sim N(\mu_i, \sigma^2).$$

Compute the MLE's of the parameters $\mu_i (i = 1, 2, \ldots, m)$ and σ^2. Show also that the null hypothesis

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_m = \mu(\text{,say})$$

is rejected whenever the test statistic $\mathcal{F}$ satisfies

$$\mathcal{F} = \frac{SS_B/(m-1)}{SS_W/(m(n-1)} > F_\alpha(m-1, m(n-1);$$

α is the significance level of the test. [4+6]

Unit-II : Marks - 20
(Theory of Chaos)

Answer ***any two*** questions.

1. (a) Convert the following equations in polar coordinate and explain the type of bifurcation arises

$$\begin{aligned}\frac{dx}{dt} &= \mu x - y + x(x^2+y^2) - x(x^2+y^2)^2 \\ \frac{dy}{dt} &= x + \mu y + y(x^2+y^2) - y(x^2+y^2)^2.\end{aligned}$$

[4]

(b) Find the fixed points of the system $\frac{dx}{dt} = \mu x + x^3$, $\frac{dy}{dt} = -y$. μ is a real parameter. Classify the fixed points and explain the bifurcation occur for this system. [3]

(c) Find the condition under which the map $\frac{dx}{dt} = x(1-x^2) - a(1-e^{-bx})$ undergoes a pitchfork Bifurcation. [3]

2. (a) Define SDIC and topological transitivity of a map. Show that the Sawtooth map attains SDIC and topological transitivity over [0,1).

[2+2+2]

(b) Obtain the Liapunov exponent of a stable periodic k-cycle. [2]

(c) Explain Sharkovskii's order and also state Sharkovskii's theorem. [2]

3. (a) Define Sierpinski tringle and Obtain its Box dimension. [2]

(b) What is the triadic expansion of a number?. Find the triadic expansions of 1, 1/3, 2/3, and 1/9. [1+2]

(c) What is meant by the self similar Dimension of any object? Find the self similar dimensions of Cantor set, Cantor dust, Peano curve,and Von Kock curve. [1+2]

(d) Define Julia Set and Mandelbrot Set. [1+1]

M. A./M. Sc. Examination-2021

Semester-III

Mathematics

MMO 31 : Optional Paper

Rings and Modules-I

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Every ring R contains unity.

Answer ***any four*** questions.

1. (a) Show that a right R-module M is simple if and only if $M =< m >$ for all nonzero $m \in M$. Hence or otherwise show that every simple module is cyclic. Does the converse hold? Justify. [4]
   (b) Define modular lattice. Let M be a right R-module. Show that $Sub(M)$ is a modular lattice. [3]
   (c) Show that the module $\mathbb{Q}_{\mathbb{Z}}$ has neither a minimal submodule nor a maximal submodule. [3]

2. (a) Let $f : M \longrightarrow N$ be an R-morphism. Prove that for every submodule X of M, $f^{-1}(f(X)) = X + \ker f$. [3]
   (b) State and prove the four lemma. [4]
   (c) Let R be a commutative ring with unity. If I and J are two ideals of R, then show that

$$0 \longrightarrow I \cap J \longrightarrow R \longrightarrow R/I \times R/J \longrightarrow R/(I+J) \longrightarrow 0$$

   is an exact sequence. [3]

3. (a) Let $(M_\alpha)_\Delta$ be a family of right R-modules. Show that any two products of this family are isomorphic. [4]
   (b) Let M be a right R-module and N be a submodule of M such that $M = \oplus_{i=1}^n M_i$ and $N = \oplus_{i=1}^n N_i$ where M_i is a submodule of M and N_i is a submodule of M_i for all i. Prove that $M/N \simeq \oplus_{i=1}^n M_i/N_i$. [3]
   (c) If $(M_\alpha)_\Delta$ is a family of right R-modules and N is a right R-module, then show that $\mathrm{Mor}_R(\oplus M_\alpha, N) \simeq \times \mathrm{Mor}_R(M_\alpha, N)$ as abelian groups. [3]

4. (a) If (F, f) is a free R-module on a nonempty set X, then show that $Im\ f$ generates F. [3]
   (b) Let R be a commutative ring with 1 and I be an ideal of R. Prove that every linearly independent subset of the R-module I has at most one element. [3]
   (c) Show that every commutative ring with 1 is an IBN-ring. [4]

5. (a) Prove that a right R-module M is Noetherian if and only if every submodule of M is finitely generated. [4]
   (b) Define composition series of a module. Prove that if a module M has a composition series then it is both Noetherian and artinian. [3]
   (c) State and prove the Fitting's Lemma. [3]

6. (a) Prove that every semisimple module is a direct sum of simple submodules. [3]
   (b) If a ring R is right semisimple then show that there is a finite family of orthogonal idempotents $e_1, e_2, \cdots, e_n$ such that $R = \oplus_{i=1}^n e_i R$. [3]
   (c) Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ and there is a right R-module L such that $M_i \simeq L$ for all $1 \leqslant i \leqslant n$. Then show that $End_R(M) \simeq M_n(End_R(L))$. [4]

M. Sc. Examination-2021

Semester-III

Mathematics

Optional Course: MMO-31(P-05) (Pure Stream)

(Algebraic Coding Theory-I)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any four* questions.

1. (a) Define a free module. Give an example of a non-free module. [1+2]
   (b) Show that the cardinalities of any two bases of a free module F over a commutative ring R with 1 are equal. Give an example of a free module in which a linearly independent subset cannot be extended to a basis. [5+2]
2. (a) Describe the addition and multiplication table of group ring $\mathbb{F}_2C_2$. Find $\mathcal{U}(\mathbb{F}_2C_2)$. [6+1]
   (b) Define augmentation map ε on a group ring RG. Show that $\varepsilon(\mathcal{U}(RG)) \subseteq \mathcal{U}(R)$. [1+2]
3. (a) Let C be a linear code of length n over F_q. Then show that $\dim C + \dim C^{\perp} = n$. Find the minimum distance of the linear code $C = \{0000, 1000, 0100, 1100\}$. [4+2]
   (b) Let $q = 2$ and $C = \{0000, 1011, 0101, 1110\}$. Decode the received word $w = 1111$. [4]
4. (a) Let $q \geq 2$ be a prime power. Then show that $B_q(n, n) = A_q(n, n) = q$. [3+3]
   (b) Find the parameters of the extended code of the linear code $C = \{000, 111, 011, 100\}$. [4]
5. (a) For an integer $q > 1$ and integers n, d such that $1 \leq d \leq n$, show that $\frac{q^n}{\sum_{i=0}^{d-1} {}^nC_i (q-1)^i} \leq A_q(n, d)$. [7]
   (b) Prove that $A_2(5, 4) = 2$. [3]
6. (a) Define $\mathrm{Ham}(r, 2)$. Show that distance of $\mathrm{Ham}(r, 2)$ is 3 for any positive integer r. [1+3]
   (b) The weight of every codeword in G_24 is a multiple of 4. [3]
   (c) Let C be a code over F_q with parameters $[n, k, d]$. Show that C is a MDS code if and only if $\mathcal{R}(C) + \delta(C) = 1$. [3]

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M. Sc. Examination-2021

Semester-III

Mathematics

Course: MMO-31 (P-02)

(Advanced Functional Analysis-I)

Time: Three Hours

Full Marks: 40

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer any four questions

1. (a) Give definition of a topological vector space (tvs) and show that every tvs is Hausdorff space. [3]

   (b) Let X be a tvs. Then prove the following:

   (i) there is a neighborhood (nbd) base of θ consisting of balanced neighborhoods (nbds).

   (ii) if X is locally convex then there exist a nbd base of θ consisting of convex and balanced nbds. [2+3]

   (c) Let X be a tvs and $A \subset X$. Show that [2]

$$\overline{A} = \cap\{\, A + V : V \text{ is any nbd of } \theta\}$$

2. (a) Let T be a non zero linear functional define over a tvs X. Show that the following statements are equivalent:

   (i) T is continuous

   (ii) Null space $N(T)$ is closed

   (iii) $N(T)$ is non-dense in X

   (iv) T is bounded in some nbd of θ in X. [8]

   (b) If X is a tvs and A, B are bounded subsets of X, then prove that $A \cup B$ and $A + B$ are bounded subsets of X. [2]

3. (a) If f is a continuous linear functional defined on a subspace M of a locally convex tvs X, then prove that there exist $T \in X^*$ such that $T = f$ on M. [3]

   (b) Let $T : (X, \tau) \to (Y, \tau')$ be a linear mapping where (X, τ) and (Y, τ') be two tvs. If T is continuous at θ then show that T is continuous on X. [3]

   (c) When a tvs X is said to be normable? Prove that X is normable if and only if θ in X has a bounded convex nbd. [4]

4. (a) State and prove closed graph theorem. [6]

   (b) Let X be a tvs. Then for $A, B \subset X$, prove that $\overline{A} + \overline{B} \subset \overline{A + B}$. [2]

   (c) Let $\mathscr{B}$ be a convex balanced local base in a tvs X. For every $V \in \mathscr{B}$, let μ_V be its Minkowski functional. Prove that $\{\, \mu_V : V \in \mathscr{B} \,\}$ is a separating family of continuous seminorms on X. [2]

5. (a) Let p be a seminorm on a vector space X. Then for $x, y \in X$, prove the following:
   (i) $p(\theta) = 0$;
   (ii) $|p(x) - p(y)| \leq p(x - y)$;
   (iii) $p(x) \geq 0$;
   (iv) $\{ x \in X : p(x) = 0 \}$ is a subspace of X;
   (v) if $B = \{ x \in X : p(x) < 1 \}$, then B is convex, balanced, absorbing and $\mu_B = p$ where μ_B denotes the Minkowski functional for B. [6]

   (b) If X is an F-space and Y is a tvs and $T_n : X \to Y$ converges pointwise to T, then prove that T is continuous. [2]

   (c) When a tvs is said to be (i) locally convex and (ii) an F-space. [2]

6. (a) Define extreme point and extreme set. [2]

   (b) State and prove Krein-Milman theorem. [6]

   (c) Let X be a tvs and Y be a Hausdorff space. If $f : X \to Y$ is continuous, then prove that the graph G of f is closed. [2]

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M. Sc. Examination-2021

Semester-III

Mathematics

Course: MMO-31 (P-02)

(Advanced Functional Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Answer any four questions

1. (a) Give definition of a topological vector space (tvs) and show that every tvs is Hausdorff space. [3]

   (b) Let X be a tvs. Then prove the following:

   (i) there is a neighborhood (nbd) base of θ consisting of balanced neighborhoods (nbds).

   (ii) if X is locally convex then there exist a nbd base of θ consisting of convex and balanced nbds. [2+3]

   (c) Let X be a tvs and $A \subset X$. Show that [2]

$$\overline{A} = \cap\{\, A + V : V \text{ is any nbd of } \theta\}$$

2. (a) Let T be a non zero linear functional define over a tvs X. Show that the following statements are equivalent:

   (i) T is continuous

   (ii) Null space $N(T)$ is closed

   (iii) $N(T)$ is non-dense in X

   (iv) T is bounded in some nbd of θ in X.

   [8]

   (b) If X is a tvs and A, B are bounded subsets of X, then prove that $A \cup B$ and $A + B$ are bounded subsets of X. [2]

3. (a) If f is a continuous linear functional defined on a subspace M of a locally convex tvs X, then prove that there exist $T \in X^*$ such that $T = f$ on M. [3]

   (b) Let $T : (X, \tau) \to (Y, \tau')$ be a linear mapping where (X, τ) and (Y, τ') be two tvs. If T is continuous at θ then show that T is continuous on X. [3]

   (c) When a tvs X is said to be normable? Prove that X is normable if and only if θ in X has a bounded convex nbd. [4]

4. (a) State and prove closed graph theorem. [6]

   (b) Let X be a tvs. Then for $A, B \subset X$, prove that $\overline{A} + \overline{B} \subset \overline{A + B}$. [2]

   (c) Let $\mathscr{B}$ be a convex balanced local base in a tvs X. For every $V \in \mathscr{B}$, let μ_V be its Minkowski functional. Prove that $\{\, \mu_V : V \in \mathscr{B} \,\}$ is a separating family of continuous seminorms on X. [2]

5. (a) Let p be a seminorm on a vector space X. Then for $x, y \in X$, prove the following:
   (i) $p(\theta) = 0$;
   (ii) $|p(x) - p(y)| \leq p(x - y)$;
   (iii) $p(x) \geq 0$;
   (iv) $\{ x \in X : p(x) = 0 \}$ is a subspace of X;
   (v) if $B = \{ x \in X : p(x) < 1 \}$, then B is convex, balanced, absorbing and $\mu_B = p$ where μ_B denotes the Minkowski functional for B.

   [6]

   (b) If X is an F-space and Y is a tvs and $T_n : X \to Y$ converges pointwise to T, then prove that T is continuous. [2]

   (c) When a tvs is said to be (i) locally convex and (ii) an F-space. [2]

6. (a) Define extreme point and extreme set. [2]

   (b) State and prove Krein-Milman theorem. [6]

   (c) Let X be a tvs and Y be a Hausdorff space. If $f : X \to Y$ is continuous, then prove that the graph G of f is closed. [2]

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M.Sc. Examination - 2021

Semester-III

Mathematics

Paper: MMO 31 (P 1)

(Advanced Complex Analysis-I)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Prove that if $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D then its imaginary part $v(x,y)$ is a harmonic function in D. [2]

   (b) Let $f(z)$ be analytic in $|z| \leq R$ and $v(r,\theta)$ be its imaginary part. Then prove that for $0 \leq r < R$ $v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)v(R,\phi)}{R^2-2rRcos(\theta-\phi)+r^2}d\phi$. [4]

   (c) Let $f(z)$ be regular in the closed disc $|z| \leq R$ and $u(r,\theta)$ be its real part. If $u(r,\theta) \geq 0$ throughout the disc, then prove that $\frac{R-r}{R+r}u(0,0) \leq u(r,\theta) \leq \frac{R+r}{R-r}u(0,0)$ $[0 \leq \theta \leq 2\pi, 0 \leq r < R]$. [4]

2. (a) Prove that Poisson's kernel $P(r,\theta;R,\phi)$ is harmonic in $|z| < R$. Further prove that $\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta;R,\phi)d\phi = 1$. [3]

   (b) Let $u(x,y)(\neq constant)$ be harmonic in a simply connected domain D. Then show that $u(x,y)$ has neither a maximum nor a minimum at any point of D. [3]

   (c) Prove that any continuous function $u(z)$ possessing the mean value property in a domain D is harmonic in D. Is converse true? [4]

3. (a) Let $f(z)$ be analytic for all finite values of z and for any $r \geq 2$,$\{M(r)\}^{\frac{1}{logr}}$ is bounded. Then show that $f(z)$ is a polynomial. [2]

   (b) If an entire function is not a polynomial, then show that its maximum modulus increases faster than the maximum modulus of any polynomial. [3]

   (c) If $f(z)$ is analytic for all finite values of z and $A(r) < K$ for all large values of r then show that $f(z)$ is constant. [2]

   (d) If $M(r)$ is the maximum value of $|f(z)|$ on $|z| = r$ then show that $logM(r)$ is a convex function of $logr$. [3]

4. (a) Prove that $max_{|z|=r}|f^n(z)| \leq \frac{2^{n+2}n!R}{(R-r)^{n+1}}\{A(R) + |f(0)|\}$. [4]

   (b) If ρ_1 and ρ_2 be the orders of the entire functions f_1 and f_2 respectively, then prove that the order of $f_1.f_2$ is at most $max\{\rho_1,\rho_2\}$. [3]

   (c) State and prove Jensen's inequality. [3]

5. (a) Define exponent of convergence of the zeros of an entire function. Show that the exponent of convergence of the zeros of $sin3z$ is 1. [3]

   (b) If $G(z)$ is a canonical product of genus $P > 0$, show that its order and exponent of convergence are same. [5]

   (c) State and prove Picard's theorem. [2]

6. (a) State Hadamard's factorization theorem. Using this theorem prove that if ρ is not an integer then $\rho_1 = \rho$. [4]

   (b) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of finite order ρ and type σ then show that $\sigma = \frac{1}{e\rho}limsup_{n\to\infty} n|a_n|^{\frac{\rho}{n}}$. [6]

5·B

Use separate answer script for each unit

M.Sc. Examination - 2021

Semester-III
Mathematics
Course: MMC-34
(Calculus of Variations and Special Functions)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I : Marks - 20)

(Calculus of Variations)
Answer ***any two*** questions.

1. (a) If $f(x) \in C[x_1, x_2]$, and if $\int_{x_1}^{x_2} f(x)g''(x)dx$ vanishes for every $g(x) \in C^2[x_1, x_2]$ such that $g(x_1) = g(x_2) = 0$ and $g'(x_1) = g'(x_2) = 0$, show that $f(x) = \alpha x + \beta$ for all $x \in [x_1, x_2]$, where α and β are constants. [3]

   (b) Find the extremal of the functional $V[y(x)] = \int_0^{\pi} (y'^2 - y^2 + 4y\cos x)dx$ with $y(0) = y(\pi) = 0$. [4]

   (c) Obtain the Euler-Ostrogradsky equation for $I[u(x,y)] = \iint_{\Omega} \{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 - 2u\phi(x,y)\}dxdy$, where $\phi(x,y)$ is known function and the values of u are prescribed on the boundary $\partial\Omega$ of the domain Ω. [3]

2. (a) Find the curves on which an extremum of the functional $J[y(x)] = \int_0^{10} y'^3 dx$, $y(0) = 0$, $y(10) = 0$ can be achieved subject to the condition that no admissible curve can pass through the interior of the circle of radius 3 having center at $(5, 0)$. [5]

   (b) Using Legendre condition, test for an extremum of the functional $I[y(x)] = \int_2^3 \sqrt[3]{x^9 y'^{-6}}\, dx$, subject to the boundary conditions determined from the relation $y(x) = x^2$. [5]

3. (a) Find the extremal of the functional $A[x(t), y(t)] = \int_{t_1}^{t_2} \frac{1}{2}(x\dot{y} - y\dot{x})dt$ subject to $\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = l$, where l is a constant. [5]

   (b) Use Galerkin method to solve the boundary value problem $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2x^2$ $(0 \le x \le 1)$ with $y(0) = 0, y(1) = 1$. [5]

Unit-II : Marks - 20)

(Special Functions)
Answer ***any two*** questions.

1. (a) Using Frobenius method, obtain the indicial equation and recurrence relation of Legendre differential equation. [3]

   (b) Using Rodrigue's formula obtain P_0, P_1, P_2 and P_3. Hence express $x^2 + x - 1$ in terms of Legendre polynomils. [2+2]

   (c) Show that Hermite polynomials are orthogonal over $(-\infty, \infty)$ with respect to the weight function e^{-x^2}. [3]

2. (a) Define regular singular point and irregular singular point of a 2nd order ordinary differential equation. How a singular point can be extended to include the point at infinity? [2+2]

   (b) Show that $J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$, $J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x)$ and $x(J_{n-1}(x) + J_{n+1}(x)) = 2nJ_n(x)$ where $J_n(x)$ is the Bessel's function of order n. [2+2+2]

3. (a) Show that $G(x,t) = \frac{1}{1-t}e^{\frac{-xt}{1-t}}$ is a generating function for Laguerre polynomial.
   Hence show that $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$
   and $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$. [3+2+2]

   (b) Show that $H_n'(x) = 2nH_{n-1}(x)$ where $H_n(x)$ is the Hermite polynomial of order n. [3]

M.A./M.Sc. Examination - 2021

Semester-III

Mathematics

Course: MMO - 31(P-11/A-13)

(Theory of Computation - I)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin

Notations and symbols have their usual meaning

Answer *any four* questions

1. (a) Define a sequential machine without output S. Show that the machine extension of the state transition from Σ_k to Σ_k^* is valid. [3]

   (b) Define the response function of a sequential machine without output S. Show that
   $$(\forall x, y, z)_{\Sigma_k^*}\ (rp_S(x) = rp_S(y)) \longrightarrow (rp_S(xz) = rp_S(yz)).$$ [3]

   (c) Define connected sub-machine of a sequential machine without output. Define homomorphism for connected sequential machines without output. Show that a mapping ϕ from a connected machine S into a connected machine T is a homomorphism if and only if
   $$(\forall x)_{\Sigma_i^*}\ \phi(rp_S(x)) = rp_T(x).$$ [4]

2. (a) Define free automaton. State the problem of minimization of free automaton. [2]

   (b) For any subset β of Σ_k^*, define R_β. Nerode's equivalence relation and show that R_β is the largest right congruence relation on Σ_k^* which refines β. Hence, define the minimal machine $M(\beta)$ associated with $F_k(\beta)$. Show that $M(\beta)$ is well-defined and its behaviour is β. [8]

3. (a) Define a probabilistic machine and the behaviour of such a machine. Let S be a finite state machine and $\bar{S}$, the probabilistic machine associated with S. Show that for any $\lambda(0 \leq \lambda < 1)$ $\beta_{\bar{S}}(\lambda) = \beta_S$. Show also that every regular set is the behaviour of some probabilistic machine but there exists a probabilistic machine S and a cut-point λ such that $\beta_S(\lambda)$ is not regular. [7]

   (b) Define isolated cut-point. State the motivation for the introduction of isolated cut-point. [3]

4. (a) Define Skolem standard form in First-order logic. Obtain the same for the form
   $$(\forall x)(\forall y)[(\exists z)(P(x,z) \wedge P(y,z)) \longrightarrow (\exists u)Q(x,y,u)].$$ [2]

   (b) Show that
   $$(\forall x)F(x) \vee (\forall x)G(x) = (\forall x)(F(x) \vee G(x)) \text{ is not a valid form.}$$ [2]

   (c) Define clausal form, semantic tree, complete semantic tree and closed semantic tree for a given set S of clauses. Show that a set S of clauses is unsatisfiable if and only if corresponding to every complete semantic tree for S there is a finite closed semantic tree. [6]

5. (a) State and prove Davis and Putnam's one-literal rule. [2]

   (b) Find an unsatisfiable set S' of ground instances of clauses in
   $$S = \{P(x), Q(x, f(x)) \vee \sim P(x), \sim Q(g(y), z)\}.$$ [2]

   (c) Define most general unifier for a given set of expressions. Write an algorithm for the unification of a given set of expressions. Show also that your algorithm computes the coveted most general unifier and terminates after a finite time. [6]

6. (a) Define resolution deduction of clauses in mechanical theorem proving. Show that a set S of clauses is unsatisfiable if and only if there is a resolution deduction of empty clause from S. [6]

(b) Use resolution deduction to examine the validity of the argument:

Students are citizens. Therefore, students' votes are citizens' votes. [4]