A.P.M
25/4/22

Use separate answer script for each unit

B.Sc. (Honours) Examination-2021

Semester-I (CBCS)
Mathematics
Course: GEC-1

(Differential Calculus, Integral Calculus & Vector Calculus)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Differential Calculus)
Answer *any four* questions.

1. State and prove Cauchy's mean value theorem. [1+4]

2. (a) State Taylor's theorem with Lagrange's form of remainder. [2]
   (b) Expand with justification the function $\sin x$ in power series about $x = 0$. [3]

3. (a) State L'Hospital's rule. Using it find the value of $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$. [1+2]
   (b) Trace the curve $a^2x^2 = y^3(2a - y)$, $a > 0$. [2]

4. Find all the asymptotes of the curve $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$. [5]

5. (a) Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are parameters connected by the relation $a^2b^3 = 3125$. [3]
   (b) Check whether $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2}$ exists or not. [2]

6. (a) State and prove Euler's theorem for the functions of two variables. [1+2]
   (b) If $V = \cos^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + \frac{1}{2}\cot V = 0$. [2]

Unit-II (Full Marks: 20)
(Integral Calculus)
Answer *any two* questions.

1. (a) If $I_{m,n} = \int \cos^m x \cos nx dx$, show that

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1} = \frac{n \sin nx \cos x - m \cos nx \sin x}{n^2 - m^2} \cos^{m-1} x - \frac{m(m-1)}{n^2 - m^2} I_{m-2,n}.$$

   Hence deduce that $\int_0^{\pi/2} \cos^n x \cos nx dx = \frac{\pi}{2^{n+1}}$. [3+2+1]

   (b) Find the areas of the surface generated by the arc of $r = 2a\cos\theta$ in the first quadrant revolving about the vertical and horizontal axes. [2+2]

2. (a) Examine whether the improper integral $\int_0^{\pi} \frac{dx}{\sin x}$ exists. If so, find its value. [2]
   (b) Test the convergence of the integral $\int_0^{\infty} \frac{x^{2s-1} - x^{2t-1}}{1-x} dx$ and specify the ranges of values of s and t. [3]
   (c) Find the length of the loop of $x = t^3$, $y = t - t^3$, where t is a parameter. [3]
   (d) Define Beta function and show that it is symmetric. [2]

3. (a) Prove that the Cauchy principal value of the integral $\int_{-1}^{1} \frac{dx}{x}$ is zero, however, the integral $\int_{-1}^{1} \frac{dx}{x}$ does not exist, in general. [2]

(b) Prove that the entire area of the loop of the curve, given by, $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)$ is $\frac{3\pi}{8}\frac{(a^2-b^2)^2}{ab}$. [4]

(c) Find the volume generated by revolving the area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$ and $y = 0$ about the x axis. [4]

Unit-III (Full Marks: 20)

(Vector Calculus)

Answer *any two* questions.

1. (a) State Stokes' theorem. Verify Stokes' theorem for $\mathbf{A} = (2x - y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. [1+3]

(b) Prove that for any vector $\mathbf{A}$, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. [3]

(c) Evaluate the integral $\int\int_S \mathbf{F} \cdot \mathbf{n} dS$, where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and S is the surface of the cube bounded by $x = 0, x = 1$; $y = 0, y = 1$; $z = 0, z = 1$. [3]

2. (a) Find the total work done in moving a particle in the force field given by $\mathbf{F} = z(\mathbf{i} + \mathbf{j}) + x\mathbf{k}$ along the helix C given by $x = \cos t$, $y = \sin t$, $z = t$ from $t = 0$ to $t = \pi/2$. [3]

(b) Suppose $\mathbf{A} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$. Evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ along the straight line joining the two points $(0, 0, 0)$ and $(1, 1, 1)$. [3]

(c) Suppose $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining any two points. Show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$. [4]

3. (a) Find the value of $\nabla\phi$ where $\phi = \frac{1}{|\mathbf{r}|}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [3]

(b) Prove that $\nabla \cdot (\phi\mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$. [3]

(c) Verify Green's theorem in the plane for $\oint_C (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. [4]

T.B

B. Sc. Honours Examination-2021

Semester-I(CBCS)

Mathematics

Core course:CC-1

(Analysis-I)

Full Marks: 60

Time: Three Hours

Questions are of values as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) Give $(\epsilon - \delta)$ definition of the limit of a function and hence evaluate $\lim_{x\to 5} f(x)$ where $f(x) = x^3 + 3x - 10,\ x \in [4, 10]$. Find δ if $\epsilon = 0.5$. [5]

   (b) If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then prove that $\lim_{x\to a} f(x)g(x) = lm$. [5]

2. (a) Prove that a function $f : [a, b] \to \mathbb{R}$ is continuous iff for any $c \in [a, b]$ and for any sequence $\{x_n\} \in [a, b]$ converging to c, $\{f(x_n)\}$ converges to $f(c)$. [5]

   (b) Find the value of $\lim_{n\to\infty} \{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\}$. [5]

3. (a) If a sequence $\{x_n\}$ converges to l then prove that the sequence $\{y_n\}$ where

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

   converges to l . [5]

   (b) Evaluate $\lim_{n\to\infty} \frac{1}{n}\{(a+1)(a+2)\cdots(a+n)\}^{\frac{1}{n}}$ if $a > 0$. [5]

4. (a) Define Cauchy sequence and give an example for the same. Prove that every Cauchy sequence is bounded. Does the converse hold? Justify. [5]

   (b) Examine if the sequence $x_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{n}$ is a Cauchy sequence. [5]

5. (a) State and Prove Lagrange's Mean Value Theorem. [5]

   (b) From the relation $f(x + h) = f(x) + hf'(x + \theta h)$, show that θ is independent on x for $f(x) = a + bx + cm^x$ where a, b, c, m are constants. [3]

   (c) If $f'(x) = 0$ for all values of x in an interval then prove that $f(x)$ is constant in that interval. [2]

6. (a) Let S be a non-empty bounded subset of $\mathbb{R}$ with $supS = M$ and $infS = m$. Find $SupT$ and $InfT$ where $T = \{x - y : x \in S, y \in S\}$. [4]

   (b) Let $x, y \in \mathbb{R}$ and $x < y$. Then prove that

   (i) there is a rational number r such that $x < r < y$.

   (ii) there is an irrational number s such that $x < s < y$.

   [3+1]

   (c) Find $SupA$ and $InfA$ where $A = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$. [2]

7. (a) Find y_n where $y = e^{ax}Cosbx$. [4]

   (b) If $y = Sin(mSin^{-1}x)$, prove that

   (i) $(1-x^2)y_2 = xy_1 - m^2y$

   (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

   [2 + 4]

8. (a) Evaluate $\lim_{x\to 0}(\frac{Sinx}{x})^{\frac{1}{x}}$. [4]

   (b) Find the values of a, b, c so that

$$\lim_{x\to 0}\frac{(a+bCosx)x - cSinx}{x^5} = 1$$

   [6]

9. (a) Obtain a reduction formula for $\int Cos^m x\ Sinnx\ dx$ and hence deduce the value of $\int_0^{\frac{\pi}{2}} Cos^5 x\ Sin3x\ dx$. [6]

   (b) A function $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and f assumes only rational values on $[0,1]$. Prove that f is a constant function. [4]

B. P

B. Sc. (Honours) Examination-2021

Semester-I (CBCS)

Mathematics

Paper: CCMA-2

(Algebra-I)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer *any six* questions.

1. (a) If $a_1, a_2, ..., a_n$ be n positive real numbers such that $a_1 \leqslant a_2 \leqslant ... \leqslant a_n$, then show that $a_1 \leqslant (a_1 a_2 ... a_n)^{1/n} \leqslant a_n$.
(b) State the Cauchy-Schwarz inequality. If a, b, c be three positive real numbers, not all equal, prove that $\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} < \frac{a+b+c}{2}$.
(c) If n is a positive integer, prove that $\frac{1}{2\sqrt{n+1}} < \frac{1.3.5...(2n-1)}{2.4.6...2n} < \frac{1}{\sqrt{2n+1}}$. [3+(1+3)+3]

2. (a) Find $|z|$ and $amp\ z$ where $z = 1 + \cos 2\theta + i \sin 2\theta$, $\frac{\pi}{2} < \theta < \pi$.
(b) State the De Moivre's theorem. Find the product of all the values of $(1+i)^{4/5}$.
(c) State and prove the remainder theorem.
(d) Find the remainder when $x^5 + x^4 - 2x^3 + 3x^2 + 5x - 5$ is divided by $(x-1)(x+2)$. [2+(1+3)+2+2]

3. (a) Solve the equation $x^4 - 6x^3 + 18x^2 - 24x + 16 = 0$ given that the equation has two roots of the form $\alpha + i\alpha$ and $\beta + i\beta$.
(b) State the Descarte's rule of signs. Apply Descarte's rule of signs to find the minimum number of complex roots of the equation $x^7 - 3x^3 + x^2 = 0$.
(c) Find the values of k for which the equation $x^3 - 12x^2 + 45x + k = 0$ may have multiple roots and solve the equation in each case. [3+(1+2)+4]

4. (a) When is a system of m linear equations in n variables called consistent? Determine whether the following system of linear equations is consistent: $y - 4z = 8$, $2x - 3y + z = 1$, $8x - 5y + 3z = 1$.
(b) Discuss the role of free variables in the uniqueness of solution of a consistent system of linear equations. Check whether the solution is unique for the following system: $3x - 4y + 2z = 0$, $-9x + 12y - 6z = 0$, $-6x + 8y - 4z = 0$. [(1+4)+(1+4)]

5. (a) Can every vector $b \in \mathbb{R}^4$ be written as a linear combination of the columns of the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{pmatrix}$? Discuss the consistency of the equation $Ax = b$, for each $b \in \mathbb{R}^4$.
(b) Find the values of c for which the vectors $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ c \\ -9 \end{pmatrix}$ are linearly dependent. [(4+1)+5]

6. (a) What is the null space N of an $n \times n$ matrix A? Show that N is a subspace of $\mathbb{R}^n$.
(b) Define column space of a matrix. Find a basis for the null sapce and a basis for the column space of the matrix $A = \begin{pmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{pmatrix}$. Also find the rank of A. [(1+2)+(1+6)]

7. (a) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $T(u) = (x+3y, 7x+5y, 3x+y)$, for $u = (x, y) \in \mathbb{R}^2$. Determine whether T is a linear transformation. Is T one-one? Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?
(b) What do you mean by the rank and nullity of a linear transformation? State Rank-Nullity theorem for a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. [(2+2+2)+(2+2)]

8. (a) When is a set $A = \{u_1, u_2, \ldots, u_k\}$ of vectors in $\mathbb{R}^n$ called an orthogonal set? If A is an orthogonal set, then show that A is a basis for a certain subspace of $\mathbb{R}^n$.
(b) Obtain an orthonormal basis for the column space of $A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$. [(1+3)+6]

9. (a) Let A be an $n \times n$ matrix with distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(1 \leq k \leq n)$. If $a_1, a_2, \ldots, a_k$ are the corresponding eigenvectors, then show that the set $S = \{a_1, a_2, \ldots, a_k\}$ is linearly independent.

If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are not all distinct, then does S still remain linearly independent? Justify.

(b) Check whether the matrix $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$ is diagonalizable and diagonalize A, if possible. [(4+2)+4]