M. Sc. Examination-2021

Semester-I

Mathematics

Course : MMC-11 (Real Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer question no. 6 and any **three** from the rest

1. (a) Show that the sets $\emptyset$ and $\mathbb{R}$ are measurable. Also find their measures. [1+1]

   (b) Find the measures of the following sets: (i) the set of irrationals in [0,2022] (ii) $\{\sin x : 0 < x < 40\pi\}$ (iii) $\{\frac{1}{n^2} : n \in \mathbb{Z}\}$ (iv)$\{\frac{1}{x^2} : 0 \leq x < \infty\}$ [1+1+1+1]

   (c) Show that Lebesgue measure is invariant under translations. [4]

   (d) Show that if $\mu^*(E) = 0$ then E is measurable. [2]

2. (a) If $B \subset A$ are measurable sets then show that $\mu(A \sim B) = \mu(A) - \mu(B)$ [2]

   (b) Prove that for any two measurable sets A and B
   $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ [3]

   (c) Suppose $\{E_n\}$ be a sequence of sets, construct a disjoint sequence of sets $\{F_n\}$ such that $\cup E_n = \cup F_n$ [2]

   (d) Prove that there exist sets that are not measurable [5]

3. (a) Show that the following are equivalent:
   i) The function f is measurable
   ii) The set $\{x \in E : f(x) > r\}$ is measurable for each rational r [4]

   (b) State and prove Egoroff theorem. [1+5]

   (c) Show that E is measurable if and only if χ_E is measurable. [2]

4. (a) Show that a Riemann integrable function is Lebesgue integral but converse is not true. [3+2]

   (b) Suppose $\{E_n\}$ be a sequence of disjoint measurable sets such that $\mu(\cup E_n)$ is finite and f is bounded and measurable function on it, then show that
   $\int_{\cup E_n} f dx = \sum_n \int_{E_n} f dx$ [5]

   (c) Evaluate $\int_0^{2\pi} f(x)dx$ where $f(x) = \begin{cases} \sin x & \text{if } x \text{ is irrational,} \\ \cos x & \text{if } x \text{ is rational} \end{cases}$ [2]

5. (a) State and prove Fatou's Lemma. Show that strict inequality may occur in it . [1+4+2]

   (b) Show that $f(x) = \frac{1}{x^{\frac{1}{3}}}$ is summable (unbounded Lebesgue integrable) in $[0, 1]$ and find $\int_0^1 f(x)dx$. [3]

   (c) If f is integrable on $[0, 1]$ then show that f is finite *a.e.* [2]

6. Answer any two

   (a) If E_1 and E_2 are measurable sets then show that $E_1 \cup E_2$ is measurable. [2]

   (b) If f is summable on E then show that $-f$ is so and $\int_E f dx = \int_E (-f)dx$ [2]

   (c) State Dominated convergence theorem and show that Bounded convergence theorem is a special case of it [2]

M. Sc. Examination - 2021

Semester-I

Mathematics

Paper: MMC 12

(Complex Analysis)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) State and prove Morera's theorem. [3]
   (b) State maximum modulus theorem. Using this theorem show that if $|f(z)| > k$ on $|z| = R, f(z)$is regular for $|z| \leq R$ and $|f(0)| < k$, then $f(z)$ has at least one zero in $|z| < R$. [3]
   (c) State and prove Taylor's theorem. [4]

2. (a) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series in the region $|z| > 3$. [3]
   (b) Show that if $f(z) = \frac{\phi(z)}{(z-\alpha)^n}$,where $\phi(z)$ is analytic at α and $\phi(\alpha) \neq 0$ then α is a pole of $f(z)$ of order n. [3]
   (c) State open mapping theorem. [1]
   (d) If β be a pole of $f(z)$ of order m and a pole of $g(z)$ of order $n(> m)$ then what kind of a point β relative to the function $f(z) + g(z)$. [3]

3. (a) If a function $f(z)$ is regular in a domain D and if $z_1, z_2, ...z_n, ...$ is a sequence of zeros of $f(z)$ having as limit point an interior point α of D, then show that $f(z) \equiv 0$. [3]
   (b) If α be a limit point of the zeros of $f(z)$ and if $f(z)$ is analytic in some deleted neighbourhood $0 < |z - \alpha| < \rho$ of α, then show that α is an isolated essential singularity of $f(z)$. [3]
   (c) Find out the zeros and determine the nature of the singularities of the function $f(z) = \frac{z+2}{z^2} sin(\frac{1}{z+1})$ and also find the residue at $z = 0$. [4]

4. (a) State and prove Casorati-Weierstrass theorem. [3]
   (b) Assume that $f(z)$ is analytic in $|z| < 1, f(0) = 0$ and $|f(z)| \leq 1$. If for a point $\alpha, 0 < |\alpha| < 1$ satisfying $|f(\alpha)| = |\alpha|$ then show that $f(z) = e^{i\theta} z(\theta = constant)$. [4]
   (c) State Cauchy's residue theorem. Using this theorem evaluate $\oint_{|z|=\frac{3}{2}} \frac{z-1}{z^2+2z} dz$. [3]

5. (a) Let $f(z)$ be analytic within and on a positively oriented simple closed curve C except for a finite number of poles within C and let $f(z) \neq 0$ any where on C, then show that
   $\frac{1}{2\pi} \Lambda_C arg f(z) = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = Z - P$,
   where Z = the number of zeros of $f(z)$ within C, a zero of order m being counted m times,
   and P = the number of poles of $f(z)$ within C, a pole of order n being counted n times. [7]
   (b) Let $f(z)$ be analytic in $|z| \leq 1$ and $|f(z)| < 1$ on $|z| = 1$. Show that there exists just one point β within the circle $|z| = 1$ such that $f(\beta) = \beta$. [3]

6. (a) If a function $f(z)$ has no singularity in the finite z-plane and has a pole of order m at infinity then $f(z)$ must be a rational integral function of degree m. [3]
   (b) Find $Res[f(z), \infty]$, where $f(z) = \frac{z}{z^2-1}$. [2]
   (c) Evaluate any one of the following by the method of contour integration:
   (i)$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$; (ii)$\int_0^{\infty} sinx^2 dx$. [5]

M. Sc. Examination-2021

Semester-I

Mathematics

MMC-13(Linear Algebra)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
(V is a finite dimensional vector space over F.)

Answer ***any four*** questions.

1. (a) Let $X, Y \in \mathbb{R}^n$ and let $A = XY^t$. Show that the characteristic polynomial of A is $t^{n-1}(t-\lambda)$ where $\lambda = X^tY$. If $\lambda \neq 0$, then prove that $E_0 \perp Y$. [2+2]

   (b) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection of $\mathbb{R}^2$ on the x-axis along the line $y = 2x$. Find two nonzero proper T-invariant subspaces of $\mathbb{R}^2$. Is there any other nonzero proper T-invariant subspaces? [3]

   (c) State and prove the Cayley-Hamilton theorem for linear operators. [3]

2. (a) Let $m(t)$ be the minimal polynomial of a linear operator $T : V \longrightarrow V$. Show that $m(t) \mid f(t)$ for every annihilating polynomial $f(t)$ of T. [2]

   (b) Give an example to show that two matrices may not be similar still they can have the same minimal polynomial. Justify. [2]

   (c) Let A be an $n \times n$ nilpotent matrix. Then show that $A^n = O$. If index of nilpotence of A is n, then show that there can not be a matrix B such that $B^2 = A$. [3]

   (d) Let $M \in M_n(F)$ and let $T : M_n(F) \longrightarrow M_n(F)$ be defined by $T(A) = MA$. Show that T and M have the same minimal polynomial. [3]

3. (a) Let $A \in M_n(F)$. Prove that eigen vectors belonging to distinct eigen values of A are linearly independent. [4]

   (b) Let A be an $n \times n$ real matrix with each entry equal to 1. Find the eigen values of A with multiplicities. Hence find a diagonal form of A. [3]

   (c) Let $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ be defined by $T(A) = A^t$. Show that 1 and -1 are the only eigen values of T. Hence or otherwise show that T is diagonalizable. [3]

4. (a) Show that a linear operator $T : V \longrightarrow V$ is triangulizable if and only if the characteristic polynomial of T splits into linear factors. [4]

   (b) If λ and μ are two distinct eigen values of a linear operator T, then show that $G_\lambda \cap G_\mu = \{\theta\}$. [3]

   (c) Find the Jordan canonical form of the linear operator $T(p(x)) = p'(x)$ on $P_2(\mathbb{R})$. [3]

5. (a) Let T be a linear operator on an inner product space V. Show that there is a unique linear operator T^* on V such that $< T(u), v >=< u, T^*(v) >$ for all $u, v \in V$. [4]

   (b) Let T be a linear operator on an inner product space V. If T has an eigen value λ then show that $\overline{\lambda}$ is an eigen value of T^*. [2]

   (c) If $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by $f(x, y, z) = 2x - 3y + 4z$, find a vector $Y \in \mathbb{R}^3$ such that $f(X) =< X, Y >$ for all $X \in \mathbb{R}^3$. [2]

   (d) Find the adjoint of the linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (x - y, 2x + y)$. [2]

6. (a) Prove that eigen vectors belonging to two distinct eigen values of a normal operator are orthogonal. [3]

   (b) Show that every eigen value of a self-adjoint linear operator is real. [2]

   (c) Prove that a linear operator T on V is unitary if and only if $\|T(v)\| = \|v\|$ for all $v \in V$. [3]

   (d) Let Q be a real skew-symmetric matrix. Then show that $I + Q$ is nonsingular. [2]

M.Sc. Examination-2021

Semester-I

Mathematics

MMC14

(Ordinary Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions out of six.

1. (a) Discuss the existence and uniqueness of the solution for the IVP.
$y' = \frac{2y}{x}$, $y(x_0) = y_0$. [4]

   (b) Find the maximal interval of existence for the unique solution of the following systems.
$x'(t) = x^2 + cos^2(t)$, $t > 0, x(0) = 0$. [3]

   (c) Apply Picard Iteration method to find first three approximation of the following IVP.
$y' = 2x(1-y)$, $y(0) = 2$.
Using Induction show that the solution tends to a unique solution. [3]

2. (a) State and Prove Sturm's Seperation Theorem. [5]

   (b) What is a Wronskian ? State Abel's Theorem. Hence show that the Wronskian is either zero everywhere or zero nowhere. [5]

3. (a) Show that the coefficient matrix A can be expressed in terms of the Fundamental Matrix $\Phi(t)$ of the system $\dot{x} = Ax$. [5]

   (b) Find the nature and stability of the fixed points of
$\dot{x} = -ax + y$
$\dot{y} = -x - ay$, for different values of the parameter a. [5]

4. (a) If N be a nilpotent matrix of order k. Then show that e^{Nt} is a series containing finite terms only. Under which condition we can write $e^{A+B} = e^A e^B$?
[1+3+1]

   (b) Consider a second-order homogeneous linear differential equation and write its adjoint equation. When is a differential equation called self-adjoint? Find the necessary and sufficient condition that a second-order linear differential equation $a_0(t)\frac{d^2x}{dt^2} + a_1(t)\frac{dx}{dt} + a_2(t)x = 0$ be self-adjoint. [1+1+3]

5. (a) Define a Sturm-Liouville Problem. Under which condition a Sturm-Liouville problem is called Regular and Singular?
Find the non-trivial solutions of the Sturm-Liouville problem $\frac{d^2y}{dx^2} + \lambda y = 0$
$y(0) = 0, y(\pi) = 0$. [1+1+4]

   (b) Find the Green's function of the BVP $y'' = 0$, $y(0) = 0 = y(1)$. [4]

6. (a) Find the solution of the system
$\dot{x} = x - 5y$,
$\dot{y} = x - 3y$
satisfying the initial condition $x(0) = 1, y(0) = 1$. Describe the behaviour of the solution as $t \to \infty$.
[4]

   (b) State Peano's Theorem for existence of first order IVP. Is the condition necessary ? Justify your answer with a suitable example. [3]

   (c) Find the 'interval of definition' for the solution of the following IVP.
$\frac{d^3x}{dt^3} + \frac{1}{t^2-4}x = cos(t)$, $x(1) = 3, x'(1) = 0, x''(1) = 0$. [3]

M. Sc. Examination-2021

Semester-I

Mathematics

Paper: MMC 15

(Partial Differential Equations)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) If $u(x,y)$ is the solution of the Cauchy problem $xu_x + u_y = 2021$, $u(x,2021) = -sin(x^2)$, $x > 0$, then find the value of $u(\sqrt{\frac{\pi}{2}}, 2021)$.
(b) Consider the second-order partial differential equation (PDE)

$$u_{xx} + 2u_{xy} + (x^2 + 4y^2)u_{yy} = log_e(x^2 + 4y^2).$$

Characterize the PDE on, inside, and outside the ellipse $x^2 + 4y^2 = 1$.
(c) Using Cauchy's method of characteristics, find an integral surface of the PDE

$$xu_x + yu_y - u_x u_y = 0$$

passing through the curve $u = 2x^2$, $y = 0$. [3+2+5]

2. (a) Find the complete integral of the PDE

$$u_{xx} - 2u_{xy} + u_{yy} - 5u_x + 5u_y + 6u = e^{2x+3y} + cos(2x + 3y).$$

(b) Solve the following PDE by Monge's method:

$$y^2 u_{xx} + 2xyu_{xy} + x^2 u_{yy} + xu_x + yu_y = 0.$$

[5+5]

3. (a) Define Monge cone of a first order nonlinear PDE $f(x,y,u,u_x,u_y) = 0$ at any point (x_0, y_0, u_0). Find the expression of the Monge cone of the PDE $(u_x - 1)(u_y - 1) = 1$ at $(0,0,0)$.
(b) Let Ω denote the unit disk given by $\{(x,y) : x^2 + y^2 \leq 1\}$ and let Ω^c be its complement in the plane. Find the number of real characteristic curves $\forall (x,y) \in \Omega^c$ of the PDE

$$(1 - y^2)u_{xx} - 2xu_{xy} + u_{yy} = 0.$$

(c) Reduce the PDE $u_{xx} + u_{xy} + u_{yy} = 0$ to canonical form. [(1+2)+2+5]

4. (a) Solve the homogeneous heat equation with the help of integral transform method

$$u_t = 4u_{xx},\ x \in (0,\pi), t > 0,$$

BC's: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,\ t > 0,$
IC: $u(x,0) = e^{2x},\ 0 \leq x \leq \pi.$

(b) Using appropriate integral transform method, solve the initial boundary value problem (IBVP) described as

$$u_{tt} = u_{xx} + sin(\pi x),\ x \in (0,1), t > 0,$$

BC's: $u(0,t) = u(1,t) = 0,\ t > 0,$
IC's: $u(x,0) = 0,\ u_t(x,0) = 0,\ 0 \leq x \leq 1.$ [5+5]

5. (a) What are the limitations of the separation of variables method related to the solution of PDE?

   (b) Consider the homogeneous Dirichlet diffusion problem

$$u_t - u_{xx} = 0,\ x \in (0,2), t > 0,$$

   BC's: $u(0,t) = u(2,t) = 0,\ t \geq 0$
   IC: $u(x,0) = x(2-x),\ x \in [0,2]$.

   Show that (i) $0 < u(x,t) < 1$ for every $t > 0$ and $0 < x < 2$ and
   (ii) $\int_0^2 u^2\, dx \leq \frac{16}{15}$ for $t \geq 0$.

   (c) With the help of separation of variables method, obtain the solution of the following initial boundary value problem (IBVP) for the wave equation:

$$u_{tt} - 9u_{xx},\ x \in (0,1), t > 0,$$

   BC's: $u(0,t) = u(1,t) = 0,\ t > 0$,
   IC's: $u(x,0) = 0,\ u_t(x,0) = x(1-x),\ 0 \leq x \leq 1$. [1+4+5]

6. (a) Define Dirichlet boundary condition for the Laplace equation in some function domain $\mathbb{R}$ and in some boundary of the domain $\partial\mathbb{R}$. If the Dirichlet problem has a solution $u(x,y)$ which is continuous in a bounded region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$, then it is unique - prove it.

   (b) Solve the following interior Neumann problem for a circle:
   PDE: $\nabla^2 u = 0$, $0 \leq r < a$, $0 \leq \theta < 2\pi$
   BC: $\frac{\partial u}{\partial r} = g(\theta)$ at $r = a$, $0 \leq \theta < 2\pi$
   where $g(\theta)$ is a continuous function on $\partial\mathbb{R}$. [(1+2)+7]

M. Sc. Examination-2021

Semester-I

Mathematics

Paper: MMC-16

(Integral Transforms and Integral Equations)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *three* from the rest.

1. Answer any *five* questions from the following: $[5 \times 2 = 10]$

   (a) Solve the Fredholm integro-differential equation
   $u'(x) = -10x + \int_{-1}^{1}(x-t)u(t)dt,\ u(0) = 1.$ [2]

   (b) Find the Laplace transform of $f(t)$, where $f(t) = cost$ when $0 < t < \pi$ and $f(t) = sint$ when $t > \pi$. [2]

   (c) Solve the homogeneous integral equation
   $y(x) = \lambda \int_0^{\frac{\pi}{2}} cosxsint\ y(t)dt.$ [2]

   (d) Using modified Adomian decomposition method, solve the integral equation
   $y(x) = e^{2x} - \frac{(e^2+1)x}{4} + \int_0^1 xt\ y(t)dt.$ [2]

   (e) Convert the integral equation $y(x) = 2x^2 + 3 + \int_0^1 k(x,t)y(t)dt$ to an equivalent differential equation, where $k(x,t) = 4t$ when $0 \le t \le x$ and $k(x,t) = 4x$ when $x \le t \le 1$. [2]

   (f) Find the inverse Laplace transform of $F(s)$, where $F(s) = \frac{s}{s^2+3s+2} + \frac{1}{(s-2)^4} + \frac{e^{-\pi s}}{s^2+1}$. [2]

   (g) With the help of an example, show that there exist functions which are neither piecewise continuous nor of exponential order but whose Laplace transform exist. [2]

   (h) Find the inverse Fourier transform of $\frac{1}{(2+is)(6+is)}$. [2]

2. (a) Find the resolvent kernel of the Fredholm integral equation
   $y(x) = e^x + \int_{-1}^{1} x^2t^2y(t)dt.$ [3]

   (b) Convert the differential equation $\frac{d^2y}{dx^2} + 4y = x, y(0) = 1, y'(1) = 0$ to an equivalent integral equation. [3]

   (c) Find the Fourier cosine transform of $f(x)$, where $f(x) = sin5x$ when $0 \le x \le 7$, and $f(x) = 0$ when $x > 7$. [2]

   (d) Find the resolvent kernel of the Volterra integral equation
   $y(x) = sinx + \int_0^x y(t)dt.$ [2]

3. (a) Using Hilbert-Schmidt theorem, solve the integral equation $y(x) = 1 + \int_0^{2\pi} sin(x+t)y(t)dt.$ [4]

   (b) If f and g are piecewise continuous on $[0,\infty)$ and of exponential order, then prove that $L\{f * g\} = L\{f\}.L\{g\}$.
   Using the above result prove that $\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ where $m, n > 0$. [2+1]

   (c) If $L\{f(t)\} = F(s)$, then prove that $L\{\int_0^t f(u)du\} = \frac{F(s)}{s}$. Hence find the Laplace transform of the integral $\int_0^t \frac{sinx}{x}dx$. [2+1]

4. (a) State Fredholm alternative theorem. Find the conditions for which the integral equation $y(x) = f(x) + \lambda \int_{-1}^{1}(x^2t^3 - t^2x^3)y(t)dt$ has infinitely many solutions. [1+3]

   (b) Using Laplace transform, solve the differential equation
   $\frac{d^2y}{dt^2} + 9y = cos2t, y(0) = 1, y(\pi/2) = -1.$ [3]

   (c) Solve, by using Laplace transform, the simultaneous differential equation
   $\frac{dx}{dt} - 2x + 3y = 0,$
   $2x + \frac{dy}{dt} - y = 0,$
   where $x(0) = 8$ and $y(0) = 3.$ [3]

5. Solve the Volterra integral equation $y(x) = x + \int_0^x (x - t)y(t)dt$ by

   (a) Laplace transform method. [2]

   (b) series solution method. [2]

   (c) Adomian decomposition method. [2]

   (d) successive approximations method. [2]

   (e) successive substitutions method. [2]

6. (a) Solve the Volterra integro-differential equation of first kind $2xe^x + e^x - x - 1 = \int_0^x (x - t + 1)u''(t)dt$, $u(0) = 0, u'(0) = 1$. [3]

   (b) Find the Fourier sine transform of e^{-ax}.
   Hence using Persaval's identity, evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx.$ [2+1]

   (c) Prove that $L\{f(t)\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-Ts}}$, where $f(t)$ is a periodic function with period T. Using the above result, find the Laplace transform of the periodic function $sint$. [2+1]

   (d) Give an example of a kernel in an integral equation that is not separable. [1]