B. Sc. (Honours) Examination-2022

Semester-I (CBCS)

Mathematics

Paper: CCMA-2

(Algebra-I)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any six*** questions.

1. (a) If a, b, c are positive real numbers and $a+b+c=1$, show that $8abc \leq (1-a)(1-b)(1-c) \leq \frac{8}{27}$.
   (b) For any positive integer n, prove that $\frac{1.3.5...(2n+1)}{2.4.6...2n} < 2\sqrt{n+1}$.
   (c) Find the maximum value of a^4b^5, where a, b are positive real numbers satisfying $5a+4b=18$. [4+3+3]

2. (a) Find the modulus and amplitude of $z = 1 + i\cot\theta$, where $0 < \theta < \pi$ and express z in polar form.
   (b) State the De Moivre's theorem. Use it to prove that $2i\sin 73\alpha = u^{73} - \frac{1}{u^{73}}$, provided that $2i\sin\alpha = u - \frac{1}{u}$.
   (c) Find all the values of $(-i)^{\frac{3}{4}}$. [4+4+2]

3. (a) Express $x^4 - 3x^3 + 10x^2$ as a polynomial in $x-3$. Also find $f(x+3)$.
   (b) Construct a biquadratic equation with real coefficients, two of whose roots are $3i \pm \sqrt{2}$.
   (c) Solve the equation $x^4 - 5x^3 - 30x^2 + 40x + 64 = 0$, provided that the roots are in geometric progression. [3+3+4]

4. (a) What do you mean by a reciprocal equation? Solve the equation $6x^4 + 35x^3 + 32x^2 + 35x + 6 = 0$.
   (b) Show that the special roots of the equation $x^{14} - 1 = 0$ are the non-real roots of $x^7 + 1 = 0$.
   (c) Solve the equation $x^3 + 3x^2 - 3 = 0$ by Cardan's method. [4+2+4]

5. (a) Find the solution(s), if there is any, of the system of linear equations $x + 3y - 5z = 4$, $x + 4y - 8z = 7$, $3x + 7y - 9z = 6$ and check if the solution is unique.
   (b) When is an indexed set of vectors $\{v_1, v_2, \ldots, v_k\}$ of $\mathbb{R}^n$ $(k < n)$ said to be linearly independent?
   Examine if the columns of the matrix $A = \begin{pmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{pmatrix}$ form a linearly independent set. [5+(1+4)]

6. (a) Find a respective basis for the null space and column space of the matrix $A = \begin{pmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{pmatrix}$.
   Also find the rank of the matrix A.
   (b) Construct a 3×3 matrix A and two non-zero vectors $b, c \in \mathbb{R}^3$ such that $b \notin \text{Col}(A)$ but $c \in \text{Nul}(A)$. [6+4]

7. (a) Define eigenvalues and eigenvectors of a square matrix A. What do you mean by eigenspace of A corresponding to an eigenvalue.
   (b) Prove that the eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_k$ of an $n \times n$ matrix A are linearly independent. Does the result hold if $\lambda_1, \lambda_2 \ldots, \lambda_k$ are not all distinct? Justify. [4+(4+2)]

8. (a) Suppose that the matrix of a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has orthonormal columns. Show that T preserves the lengths and orthogonality of vectors.
   (b) Use Gram-Schmidt process to obtain an orthonormal basis for the column space of the matrix

$$\begin{pmatrix} -10 & 13 & 7 & -11 \\ 2 & 1 & -5 & 3 \\ -6 & 3 & 13 & -3 \\ 16 & -16 & -2 & 5 \\ 2 & 1 & -5 & -7 \end{pmatrix}.$$

9. (a) Let λ be an eigenvalue of an $n \times n$ orthogonal matrix A. Show that $\frac{1}{\lambda}$ is an eigenvalue of A^T. [4+6]
   (b) Diagonalize the matrix $\begin{pmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{pmatrix}$, if possible. [4+6]

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B. Sc. Honours Examination-2022

Semester-I(CBCS)
Mathematics
Core Course: CC-1
(Analysis-I)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer any six questions

1. (a) Evaluate $\lim_{x\to a} \cos x$ by using $(\epsilon - \delta)$ definition of the limit of a function. [2]

   (b) If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m(\neq 0)$ then prove that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$. [6]

   (c) Examine if $\lim_{x\to 0} \frac{2x+|x|}{2x-|x|}$ exists. [2]

2. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous such that for $c \in (a, b)$, $f(c) \neq 0$. Show that $f(x)$ has the same sign as that of $f(c)$ in some neighbourhood of c. [4]

   (b) A function f is defined by

   $$f(x) = \begin{cases} \cos\frac{1}{x} & for\ x \neq 0 \\ 0 & for\ x = 0 \end{cases}$$

   Is f continuous at $x = 0$? Support your answer. [3]

   (c) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at one point $c \in \mathbb{R}$, then prove that f is continuous at every point in $\mathbb{R}$. [3]

3. (a) State and prove nested intervals theorem. [5]

   (b) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \geq 1$ converges to 2. [3]

   (c) Evaluate $\lim_{n\to\infty} \frac{1.3.5.\cdots.(2n-1)}{2.4.6.\cdots.(2n)}$. [2]

4. (a) Prove that a sequence of real numbers is convergent iff it is a Cauchy sequence. [6]

   (b) Give an example of a sequence of rational numbers that converges to an irrational number. [1]

   (c) Evaluate $\lim_{n\to\infty} \frac{1}{n} \{(2n+1)(2n+2)\cdots(2n+n)\}^{\frac{1}{n}}$. [3]

5. (a) Prove that a point x is a limit point of a set A in $\mathbb{R}$ iff every neighbourhood of x contains infinitely many points of A. [3]

   (b) Let S be a non-empty bounded subset of $\mathbb{R}$ and $T = \{-x : x \in S\}$. Prove that T is bounded and $\sup T = -\inf S$ and $\inf T = -\sup S$. [4]

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(c) State Archimedean property of $\mathbb{R}$. Prove that if $x \in \mathbb{R}$ and $x > 0$ then there exists a natural number n such that $0 < \frac{1}{n} < x$. [3]

6. (a) If $y = x^{n-1}logx$ then find y_n. [3]

(b) If $y = (x^2 - 1)^n$ then show that

$$(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

[4]

(c) If $y = x^{2n}$ where n is a positive integer, then show that

$$y_n = 2^n\{1.3.5.\cdots.(2n-1)\}x^n.$$

[3]

7. (a) State and prove Rolle's theorem and give its geometrical interpretation. [5]

(b) Prove that $\frac{x}{x+1} < log(1+x) < x, \quad \forall\ x > 0.$ [3]

(c) In the Mean Value Theorem, $f(x+h) = f(x)+hf'(x+\theta h)$, show that θ is independent of x where $f(x)$ is given by $f(x) = a + bx + cm^x$. [2]

8. (a) State L. Hospital's theorem and hence evaluate $\lim\limits_{x\to 0} (\frac{\tan x}{x})^{\frac{1}{x}}$. [5]

(b) Find the constants a and b so that

$$\lim_{\theta\to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1.$$

[5]

9. (a) If $I_n = \int x^n\sqrt{a-x}\ dx$ then prove that $I_n = \frac{2na}{2n+3}\ I_{n-1} - \frac{2x^n}{2n+3}(a-x)^{\frac{3}{2}}$ and hence evaluate $\int_0^a x^3\sqrt{ax-x^2}\ dx$. [6]

(b) State and prove Cauchy's Mean Value theorem and deduce Lagrange's Mean Value theorem from this theorem. [4]

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