B.SC sem-II 2023

Use separate answer scripts for each unit

B.Sc.(Honours) Examination, 2023

Semester-II (CBCS)
Mathematics (Honours)
Core Course : CC-3
(Analysis-II and Algebra-II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meaning.

Unit-I [Analysis-II (Marks-30)]
Answer ***Question no.1*** and ***any four*** from the rest

1. Answer any five questions : [2 × 5]
   (a) Find closure of the sets $\mathbb{N}$, $\mathbb{R}-\mathbb{Q}$, $\mathbb{Q}-\mathbb{Z}$, and $\{\frac{1}{n} : n = 1, 2, \cdots\}$.
   (b) Find the derived set of the set $\bigcup_{i=0}^{\infty}[(i, i+1) - \{i+\frac{1}{3}, i+\frac{2}{3}\}]$.
   (c) Give examples of countable, uncountable and finite open covers of $\mathbb{R}$.
   (d) Show that $\mathbb{N}$ is a closed set but $\mathbb{Q}$ is neither open nor closed.
   (e) Find total length of the curve $x^2 + y^2 + 2x = 0$.
   (f) Find the convergent subsequences of the sequence $\{\sin\frac{n\pi}{2}\}$.
   (g) Find asymptote of the curve $y^3 + x^3 = 3xy$.
2. State and prove Taylor's theorem with Cauchy's form of remainder. [5]
3. Find the area inside the two cardioids $r = 5(1+\cos\theta)$ and $r = 5(1-\cos\theta)$. [5]
4. Expand $(1+x)^m$, $m \in \mathbb{R} \sim \mathbb{N}$, in power series with proper justification [5]
5. Define compact sets. State and prove Heine-Borel covering theorem. [5]
6. (a) Show that a point x is limit point of S if and only if there is a sequence $x_n \in S \sim \{x\}$ which converges to x. [3]
   (b) What is nested intervals. State Nested interval theorem [2]
7. (a) Find the intrinsic equation of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - cos\theta)$, the fixed point being the origin. [3]
   (b) Find envelope of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters are connected by the relation $a + b = 1$ [2]

Unit-II (Marks: 30)
(ALgebra-II)
Answer ***all the*** questions.

8. Answer ***any two*** questions.$(2 \times 5 = 10)$
   (a) i. Show that no integer in the following sequence is a perfect square: 11,111,1111,11111,.... [2]
   ii. For $n \geq 1$, and positive integers a, b if $a^n | b^n$ find GCD(a, b). [3]
   (b) i. For a composite number $n > 4$, prove or disprove $n|(n-1)!$ [3]
   ii. How many primes are there atleast less than 2^{1024}? [1]
   iii. State the 'Fundamental Theorem of Arithmetic'. [1]

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(c) i. Find the last two digit of 9^{9^9}. [2]

ii. Establish the congruence $2222^{5555} + 5555^{2222} \equiv 0(\text{modulo } 7)$. [3]

9. Answer ***any two*** questions.$(2 \times 10 = 20)$

(a) i. Describe explicitly the sets $K = \{a^2 \mid a \in D_4\}$ and $L = \{a \in D_4 \mid a^2 = e\}$. Are K and L subgroups of D_4? If so, which of them are commutative as a group. [2+1+1]

ii. Let G be a finite group of even order. Show that G contains exactly odd number of elements of order 2. [3]

iii. Show that a semigroup S is a group if and only if for all $a, b \in S$, the equations $ax = b$ and $ya = b$ have solutions in S. [3]

(b) i. If H and K are two nontrivial subgroups of $\mathbb{Q}$, then show that $H \cap K$ is also nontrivial. Does this hold if $\mathbb{Q}$ is replaced by $\mathbb{R}$? Justify. [2+1]

ii. Let H be a subgroup of G, and $a \in G$ such that $a^n = e$ and $a^m \in H$. If $\gcd(m, n) = 1$, then show that $a \in H$. [3]

iii. Let H and K be two subgroups of a group G. Prove that HK is a subgroup if and only if $HK = KH$. Consider the subgroups $H = \{e, (12)\}$ and $K = \{e, (123), (132)\}$ of S_3. Is HK a subgroup of S_3? [3+1]

(c) i. Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic. Find two generators of the subgroup $H =< \frac{2}{3}, \frac{4}{5} >$ of $\mathbb{Q}$. Is there any other generators of H? [2+1+1]

ii. Let G be a finite cyclic group of order n. If d is a positive integer such that $d \mid n$, show that G has a unique subgroup of order d. [3]

iii. Let G be a cyclic group of order 450. Find total number of distinct subgroups and generators of G. How many elements of G are of order 50? [3]

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B.Sc. (Honours) Examination, 2023

Semester-II (CBCS)
Mathematics
Course: CCMA-4
(Geometry & Vector Calculus)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
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Unit-I (Full Marks: 30)

(Geometry)

Answer **Question no. 1** and ***any four*** from the rest.

1. Answer any **five** questions: [5 × 2=10]

   (a) Determine the transformed equation of $11x^2+16xy-y^2=0$ when the axes are rotated through an angle $\tan^{-1}\frac{1}{2}$?

   (b) Determine the value of λ and μ so that $\lambda x^2+3xy-2y^2-5x+\mu=0$ represents a pair of perpendicular lines.

   (c) Find the values of a and f for which $ax^2-6xy+9y^2+4x+2fy+1=0$ may represent a conic with infinitely many centres.

   (d) Find the equation of the pair of straight lines joining the origin to the points of intersection of the ellipse $2x^2+4y^2=8$ and the line $2x+y=3$.

   (e) What do you mean by 'invariant'? Give an example.

   (f) Find the length of the common chord of the circles $x^2+y^2-4x+2y+1=0$ and $x^2+y^2+2x-4y+1=0$.

   (g) If $px^2+5y^2+qz^2-axy+2px-4qy-2az+d=0$ represents a sphere of radius 2, then find the value of d.

   (h) Find the equation of the cone whose vertex is the origin and base is the circle $x=a,\ y^2+z^2=b^2$.

2. If by rotation of transformation of rectangular axes, the expression $\frac{x^2}{\lambda}+\frac{y^2}{\mu}$ transforms to [5]

   $px'^2+2rx'y'+qy'^2$, then show that the expression $\frac{x^2}{a-k}+\frac{y^2}{b-k}$ transforms to

   $$\frac{px'^2+2rx'y'+qy'^2+k(r^2-pq)(x'^2+y'^2)}{1-(p+q)k-(r^2-pq)k^2}.$$

3. Show that one of the bisectors of the angles between the pair of lines $ax^2+2hxy+by^2=0$ will pass through the point of intersection of the two lines $ax^2+2hxy+by^2+2fx+2gy+c=0$, if $h(f^2-g^2)=fg(a-b)$. [5]

4. Reduce the equation $x^2-2xy+y^2+6x-14y+29=0$, to its canonical form and determine the nature of the conic. Also find the equation of its axes. [5]

5. Determine the limiting points of the coaxial system of circles given by $S=x^2+y^2+2x-4y+5=0$ and $S'=x^2+y^2-4x+6y+13=0$. [5]

6. A sphere of radius r passes through the origin O and cuts the axes in P, Q, R respectively. Find the locus of the centroid of the triangle PQR. [5]

7. Find the equation of the right circular cone whose vertex is at $(-1, 2, 3)$, the semi-vertical angle is 60° and the axis is parallel to $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$. [5]

Unit-II (Full Marks: 30)

(Vector Calculus)

Answer **Question no. 1** and ***any four*** from the rest.

1. Answer any **five** questions: [5 × 2=10]

   (a) If $\vec{\rho}_1$ and $\vec{\rho}_2$ are the unit vectors inclined at an angle θ, then show that $2\sin\frac{\theta}{2} = |\vec{\rho}_1 - \vec{\rho}_2|$.

   (b) If $\vec{a} = \hat{i} - 2\hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, then find the component of $\vec{a}$ along $\vec{b}$.

   (c) Find the volume of the tetrahedron, the co-ordinates of whose vertices are $(0, 1, 2)$, $(3, 0, 1)$, $(1, 1, 1)$ and $(4, 3, 2)$.

   (d) Find the work done by the forces $4\hat{i} + \hat{j} - 3\hat{k}$, $3\hat{i} + \hat{j} - \hat{k}$ and $\hat{i} - 3\hat{k}$ which together work on a particle and displace it from the point $(1, 2, 3)$ to the point $(5, 4, 1)$.

   (e) Find the maximum value of the directional derivative of $\phi = x^2 + z^2 - y^2$ at the point $(1, 3, 2)$.

   (f) Define curl and divergence of a vector quantity.

   (g) If $\vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + bt\hat{k}$, then find $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$.

   (h) State Serret-Frenet formulae.

2. (a) Use vector method to show that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ in a triangle ABC. [3]

   (b) Show that $|\vec{a} \times \vec{b}|^2|\vec{a} \times \vec{c}|^2 - \{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})\}^2 = |\vec{a}|^2[\vec{a}\ \vec{b}\ \vec{c}]^2$. [2]

3. A space curve is given by $x = t$, $y = t^2$, $z = t^3$. Find the curvature (κ) and torsion (τ) at $t = 0$. [5]

4. (a) State and prove a necessary and sufficient condition for a vector function $\vec{r} = \vec{g}(t)$ to have a constant magnitude. [3]

   (b) If $\frac{d\vec{a}}{dt} = \vec{r} \times \vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{r} \times \vec{b}$, then show that $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{r} \times (\vec{a} \times \vec{b})$, where $\vec{r}$ is a constant vector and $\vec{a}$, $\vec{b}$ are vector functions of a scalar variable t. [2]

5. Show that the vector $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. For $\vec{F}$, find a scalar function ϕ such that $\vec{F} = \vec{\nabla}\phi$. [2+3]

6. If $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{H} = 0$, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}$ and $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$, then show that $\nabla^2\vec{H} = \frac{\partial^2\vec{H}}{\partial t^2}$ and $\nabla^2\vec{E} = \frac{\partial^2\vec{E}}{\partial t^2}$. [5]

7. (a) Let $\vec{a}$ and $\vec{b}$ two vector functions. Prove that $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot (\text{curl}\ \vec{a}) - \vec{a} \cdot (\text{curl}\ \vec{b})$. [3]

   (b) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$. [2]