Use separate answer scripts for each unit

B.Sc. (Honours) Examination 2023

Semester-II (CBCS)
Mathematics
Paper: CC-11
(Analysis-V and Differential Equation-III)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meaning.

Unit-I (Analysis-V)
(Full Marks: 30)]
Answer *any three* questions

1. (a) If ϕ is bounded and monotonic in $[a,\infty]$ and $\int_a^\infty f\,dx$ is convergent then show that $\int_a^\infty f\phi\,dx$ is convergent. [4]

   (b) Show that $\int_0^\infty \frac{\sin x}{x}$ is convergent but not absolutely. [5]

   (c) Prove that $\int_1^\infty \frac{\sin x}{x^p}$ is absolutely convergent for $p>1$. [1]

2. (a) Let $\{f_n\}$ be a sequence of R-integrable functions converges to f uniformly on $[a,b]$. Show that f is R-integrable on $[a,b]$ and $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$. Also show that the condition of uniform convergence can not be omitted from above statement. [4+2]

   (b) Consider the sequence $\{f_n\}$,

   $$f_n(x)=\begin{cases}1-nx & \text{for } 0\le x\le \frac{1}{n},\\ 0 & \text{for } \frac{1}{n}<x\le 1.\end{cases}$$

   Find its limit function f and show that $\lim_{n\to\infty}\int_0^1 f_n = \int_0^1 f$. Is the convergence of $\{f_n\}$ uniform? [3]

   (c) Show that the convergence of $\{x^n\}$ on $[0,1]$ is not uniform. [1]

3. (a) State and prove Dirichlet's test for the uniform convergence of the series $\sum f_n g_n$. [4]

   (b) Consider the series $\sum \frac{\sin nx}{n^p}$ with $0\le p\le 1$ in the interval $[a,b]$.

   i) Show that the series converges uniformly in $[a,b]$ if $2k\pi < a < b < 2(k+1)\pi$, $k\in\mathbb{Z}$.

   ii) Show that the convergence is not uniform in any interval which includes the points $2k\pi$, $k\in\mathbb{Z}$. [3+3]

4. (a) State and prove Weierstrass's Approximation Theorem. [5]

   (b) Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{x^n}{n^2 2^n}$. [2]

   (c) If $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} b_n x^n$ for some non zero distinct points $c_1, c_2, \ldots$ with $c_k\to 0$ then show that the two power series are identical. [3]

5. (a) When is the trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ referred to as a Fourier series?

   Show that if $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ converges uniformly in $[-\pi,\pi]$ to a function f then it is the Fourier series of f. [1+3]

   (b) If a_n and b_n are Fourier coefficient of f then show that $a_n, b_n\to 0$ as $n\to\infty$. [4]

   (c) Show that the Fourier series of x^2 in $[-\pi,\pi]$ is $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\left[\frac{\cos nx}{n^2}\right]$. [2]

Unit-II (Differential Equations-III)

(Full Marks: 30)

Answer question number 1 and ***any two*** from the rest.

1. Answer ***any five*** questions.

(a) Find the partial differential equation (PDE) for the family of planes if the intercepts made by the arbitrary member of the family on the co-ordinate axes are a, b and 1 ($a \neq 0$, $b \neq 0$).

(b) Find the general integral of the PDE: $\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 3x + 4y + 5z$.

(c) Prove that the following PDEs are compatible. $z^2 = xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$, $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$.

(d) Find the family of surfaces orthogonal to the family of surfaces whose PDE is given by $(y+z)\frac{\partial z}{\partial x} + (z+x)\frac{\partial z}{\partial y} = x + y$.

(e) Let $z = z(x, y)$ be the complete integral of the PDE $pq = x^2y^3$ passing through the points $(0, 0, 4)$ and $(3, 2, 16)$ in the xyz-space. Find the value of $z(x, y)$ at $(-3, 4)$.

(f) Show that there is no singular integral for a PDE of the following type: $f\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$.

(g) Find the initial strip of the following PDE passing through the curve $\Gamma : x^2 = z,\ y = 0$ in the context of Cauchy's method of characteristic:

$$\left(\frac{\partial z}{\partial x}\right)^2 - 3\left(\frac{\partial z}{\partial y}\right)^2 = z.$$

(h) Find the complementary function (CF) of the following PDE:

$$\left(D^3D' + D^2D'^2 - 6DD'^3\right) z = y\cos(3y - 2x);\ D^n \equiv \frac{\partial^n}{\partial x^n},\ D'^n \equiv \frac{\partial^n}{\partial y^n}.$$

[5 × 2=10]

2. (a) Find the general integral and integral surface passing through the curve $3x = 2y = z$ of the following PDE:

$$x(y^2 + z^2)\frac{\partial z}{\partial x} + y(x^2 + z^2)\frac{\partial z}{\partial y} = (x^2 - y^2)z.$$

(b) Let $z(x, y)$ be the solution of the following Cauchy problem: $\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} = 1 - z;\ -\infty < x < \infty,\ y \geq 0,\ z(x, 0) = \cos(x)$. Find the value of $z(0, 1)$ and hence comment on the nature of the solution. [5+5]

3. (a) Find the complete integral of the following first-order nonlinear PDE by the Charpit's method.

$$\frac{\partial z}{\partial x} = \left(y\frac{\partial z}{\partial y} + z\right)^2.$$

Hence find the singular integral, if any.

(b) Solve:

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} + x - y.$$

[(4+2)+4]

4. (a) Classify the compatible systems of first-order PDEs in two independent variables. Find a one-parameter family of common solutions of the following PDEs: $x\left(\frac{\partial z}{\partial x}\right)^2 + y\left(\frac{\partial z}{\partial y}\right)^2 = z,\ x\left(\frac{\partial z}{\partial x}\right)^2 = y\left(\frac{\partial z}{\partial y}\right)^2$.

(b) Find a surface passing through the parabolas, given by $z = 0,\ y^2 = 4\alpha x$ and $z = 1,\ y^2 = -4\alpha x$ and satisfying the PDE:

$$x\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} = 0.$$

[(2+3)+5]

B.Sc. (Honours) Examination, 2023
Semester-V (CBCS)
Mathematics
Paper: CCMA-12
(Numerical Analysis)

Time: Three Hours | Full Marks: 60

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***question no. 1*** and ***any five*** from the rest.

1. Answer any **five** questions: [5 × 2=10]

(a) Distinguish between limiting absolute error and limiting relative error.

(b) State Weierstrass Approximation Theorem.

(c) Taking $h = 2$, what will be the numerical value of $\Delta^{2024}(2024x^{2024} - 2023x^{2023} + 2022)$?

(d) Show that $K_i^{(n)} = K_{n-i}^{(n)}, i = 0(1)n$.

(e) State one advantage and one disadvantage of Graeffe's root squaring method.

(f) Write down the equation $x^3 + 2x - 10 = 0$ in the form of $x = \phi(x)$ such that the iterative scheme about $x = 2$ converges.

(g) Find all $\alpha > 0$ and $\beta > 0$ so that the following matrix is strictly diagonally dominant.

$$M = \begin{bmatrix} 4 & \alpha & 1 \\ 2\beta & 5 & 4 \\ \beta & 2 & \alpha \end{bmatrix}$$

(h) What do you mean by single step method? Explain with a suitable example.

2. (a) What do you mean by rounding off an approximate number? State the rules of rounding off a number with suitable examples. [1+4]

(b) Find absolute, relative and percentage error in computing $f(x) = 3\sin x - 2x^2 - 9$ for $x = 0$ when the error in x is 0.003. [3]

(c) Find the product of the approximate numbers $1.3719, 29.086, 88.61, 36.1993, 11.8$, which are correct upto the indicated number of digits. Give proper justification in support of your answer. [2]

3. (a) Find the missing terms in the following table:

x	0	1	2	3	4	5
$f(x)$	0	?	8	15	?	35

[4]

(b) Prove that $1 + \delta^2\mu^2 \equiv \left(1 + \frac{\delta^2}{2}\right)^2$ [3]

(c) If $f(x)$ be a polynomial of degree three and $f(0) = -1, f(1) = 5, f(2) = 13, f(3) = 36, f(4) = 69$, where $f(2)$ is not correct, then find the error in $f(2)$. [3]

4. (a) Establish Newton's backward interpolation formula when the functional values $y = f(x)$ are known at $(n + 1)$ points. Also derive the error involved in this formula [6]

(b) Using the following data, find the value of $x(1.2461)$, correct to two decimal places. [4]

x:	0.0	0.1	0.2	0.3	0.4
y:	1.0000	1.1052	1.2214	1.3459	1.4919

5. (a) A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod are given in the following table for various values of the time t (in second).

t (sec)	0	0.1	0.2	0.3	0.4	0.5	0.6
x (cm)	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Find the velocity and acceleration of the slider when $t = 0.6$. [4]

(b) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then show that g has a fixed point in $[a,b]$. [3]

(c) Determine the step size that can be used in the tabulation of $f(x) = \sin x$ in the interval $\left[0, \frac{\pi}{4}\right]$ at equally spaced nodes so that the maximum error in the quadratic interpolation is less than 5×10^{-8}. [3]

6. (a) Estimate the error involved in Simpson's $\frac{1}{3}$rd rule. Explain geometrically why does this rule call a parabolic rule. [4+2]

(b) A river is 80 ft wide. The depth d in feet at a distance x ft from one bank is given in the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Estimate the area of the cross-section of the river. [4]

7. (a) Derive the Newton-Raphson method to determine approximately one real root of an equation $f(x) = 0$. Give its geometric interpretation and discuss its convergence. [2+2+4]

(b) Find the number of iterations needed in the bisection method to obtain an approximation with accuracy 10^{-5} to the solution of $x^3 + 4x^2 - 10 = 0$ lying in the interval $[1,2]$. [2]

8. (a) Carry out an error analysis to show the convergence of Gauss-Seidel iterative method for solving a system of n linear non-homogeneous equations involving n unknowns. [5]

(b) Compute $y(0.4)$, from the initial value problem $\frac{dy}{dx} = \sqrt{x^2 + y}$, $y(0) = 0.8$, taking $h = 0.2$, by Runge-Kutta method of fourth order, correct to four decimal places. [5]

9. (a) Derive the Modified Euler's method for solving the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. [5]

(b) Solve the following system of equations

$$4.21x_1 + 22.67x_2 + 3.85x_3 = 30.24$$
$$2.31x_1 + 31.49x_2 + 1.52x_3 = 40.60$$
$$3.49x_1 + 4.85x_2 + 28.97x_3 = 42.81$$

using Gaussian elimination method with partial pivoting, correct to four decimal places. [5]

Use separate answer script for each unit

B.Sc. (Honours) Examination-2023

Semester-V (CBCS)
Mathematics
Course: DSEMA-1
(Mechanics-II)

Time: Three Hours

Full Marks: 60

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 40)

(Dynamics of a rigid body)

Answer *any four* questions.

1. (a) Obtain the moment of inertia (M.I.) of a solid sphere about a diameter. [5]
   (b) Show that the M.I. of an elliptic area of mass M and semi-axes a and b about a diameter of length r is $M\frac{a^2b^2}{4r^2}$. [5]

2. (a) Find the condition for a straight line, at any point of its length, to be a principal axis of a material system. [5]
   (b) A uniform lamina is bounded by a parabolic arc of latus rectum $4a$ and a double ordinate at a distance b from the vertex. If $b = \frac{a}{3}(7 + 4\sqrt{7})$, then show that two of the principal axes at the end of the latus rectum are the tangent and the normal there. [5]

3. (a) Obtain the equation of motion of a rigid body relative to the centre of inertia. [5]
   (b) A rough uniform board of mass M and length $2l$, rests on a smooth horizontal plane, and a man of mass m walks on it from one end to the other. Show that the distance through which the board moves is

$$\frac{2ml}{m+M}.$$

[5]

4. (a) Obtain kinetic energy of a rigid body rotating about an axis. [3]
   (b) A fine string has two masses M and M' tied to its ends and passes over a rough pulley, of mass m and radius a, whose centre is fixed. If the string does not slip over the pulley, then show that M will descend with an acceleration equal to

$$\frac{(M-M')a^2}{(M+M')a^2+mk^2}g,$$

   where k is the radius of gyration of the pulley. [7]

5. (a) Define compound pendulum. Prove that the centres of oscillation and suspension are interchangeable. [1+4]
   (b) An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes thorugh the centre of oscillation. Prove that the eccentricity of the ellipse is $\frac{1}{2}$. [5]

6. (a) Obtain the kinetic energy of a body moving in two dimensions. [5]
   (b) A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through their axes moves like a circular pendulum of length

$$(a-b)\left(\frac{k^2}{a^2}\right).$$

[5]

Unit-II (Full Marks: 20)

(Hydrostatics)

Answer ***any two*** questions.

1. (a) What do you mean by a 'homogeneous fluid'? Show that in a homogeneous fluid at rest under gravity, the difference of pressures between any two points is proportional to the difference of their depths. [1+4]

   (b) A fine glass tube in the form of an equilateral triangle is filled with equal volumes of three liquids which do not mix and whose densities are in arithmetic progression. The tube is held in a vertical plane, and the side that contains portions of the heaviest and lightest liquids make an angle α with the vertical. Show that the surface separation divides the side in the ratio $\cos(30^\circ - \alpha) : \cos(30^\circ + \alpha)$. [5]

2. (a) A homogeneous liquid of density ρ is in equilibrium under the forces $\frac{\mu}{x}, \frac{\mu y}{y^2+z^2}, \frac{\mu z}{y^2+z^2}$ (μ is a constant) at (x, y, z). Find the equation of the free surface. Also, show that pressure at the point (x, y, z) is $\Pi + \rho\mu \log \sqrt{x^2(y^2+z^2)}$, where Π is the atmospheric pressure. [4+1]

   (b) A hollow weightless cone of semi-vertical angle α and of height h is filled with liquid of density ρ and freely hung from a point on the rim of its base. Show that the thrust on the base is $\frac{4\pi\rho g h^3 \tan^3\alpha \sin\alpha}{\sqrt{1+15\sin^2\alpha}}$. [5]

3. (a) A cylindrical tumbler, half filled with a liquid of density ρ', is filled up with a liquid of density ρ, which does not mix with the former one. Show that the ratio of the thrust on the base of the tumbler to the whole pressure on its curved surface is $2r(\rho+\rho') : h(\rho'+3\rho)$, where h is the height of the tumbler and r the radius of its base. [5]

   (b) Define centre of pressure of a given plane area in a fluid in equilibrium. Show that the depth of centre of pressure (C.P.) of a plane area immersed in a homogeneous liquid is greater than the depth of its centroid or centre of gravity (C.G.). [1+4]

Use separate answer script for each unit

B.Sc. (Honours) Examination-2023

Semester-V (CBCS)
Mathematics
Paper: DSE-2
(Linear Programming Problem, Game Theory and Mathematical Statistics)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Linear Programming Problem and Game Theory)

Answer ***any five*** questions.

1. A firm can produce three types of woolen clothes: A, B, and C. Three kinds of wool, say R, G, and Y, are required. One unit length of type A cloth needs 3 yards of R-type wool and 2 yards of Y-type wool; one unit of B-type cloth needs 2 yards of R-type wool and 2 yards of G-type wool, while one unit length of type C cloth needs 6 yards of R-type wool and 4 yards of Y-type wool. The firm has only 18 yards of R-type, 15 yards of G-type, and 21 yards of Y-type wool. It is assumed that the profit obtained from one unit length of type A cloth is Rs.3, of type B cloth is Rs.5, and that of type C cloth is Rs.2. Formulate a LPP to determine how the firm should utilize available materials mentioned above to maximize the profit from the finished cloth. Solve it. [3+3]

2. Discuss the difference between linear and convex combinations of two vectors in $\mathbb{R}^n$ $(n \geq 2)$ with illustrations. Define convex set. Give one example of an unbounded convex set. Examine whether the set $X = \{(x_1, x_2) | x_1 x_2 \leq 4\}$ is a convex set. [2+1+1+2]

3. Define convex hull, separating and supporting hyperplanes of a set. Prove that the optimal solution of a LPP lies on a supporting hyperplane of the convex set of feasible solutions to the LPP. Verify this statement for the following LPP [1+3+2]

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 3, \\ & 3x_1 + x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

4. Rewrite the following LPP

$$\begin{aligned} \text{Maximize} \quad & Z = 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 2, \\ & 4x_1 + 3x_2 \geq 6, \\ & x_1, x_2 \geq 0 \end{aligned}$$

in standard form by incorporating additional variables. Indicate the types of the variables precisely. Solve this LPP by using (Charne's) Big-M method. [2+4]

5. Give a schematic diagram for the interchange between primal and dual problems. Prove that dual of the dual is the primal. Use the principle of duality to solve the following LPP

$$\begin{aligned} \text{Maximize} \quad & Z = 5x_1 + 12x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 5, \\ & 2x_1 - x_2 \quad\quad \geq 6, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

by the graphical method, if admissible. [1+3+2]

6. Use the MODI method to find the minimum cost for the transportation problem: [6]

	D_1	D_2	D_3	a_i
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
b_j	4	2	2	

7. Express an assignment problem in the form of an LPP. Find the optimal assignment to find the minimum cost for one of the following two assignment problems [3+3]

(a)

	J_1	J_2	J_3
P_1	120	100	80
P_2	80	90	110
P_3	110	140	120

(b)

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7
D	5	7	7	6

8. Explain games with pure and mixed strategies. Solve a game with the following payoff matrix [3+3]

	A	B	C	D	E
I	3	6	4	9	6
II	5	6	1	7	8
III	8	7	9	8	7
IV	4	2	8	5	1

Unit-II (Mathematical Statistics)

Answer ***any three*** questions.

1. (a) Define a statistic and sampling distribution of a statistic corresponding to the population of a random variable X. Distinguish between the sampling distribution of a statistic and distribution of a sample. [1+1+2]

(b) Find the sampling distribution of the mean for the Poisson population with parameter μ. [2]

(c) Show that the sample mean $\overline{X}$ and the Sample variance S^2 are uncorrelated if the third-order central moment (μ_3) vanishes. [4]

2. (a) Define an estimate of a population parameter. When is an estimate said to be consistent and unbiased? Find a consistent and unbiased estimate of the parameter σ^2 of a normal (m, σ) population. [1+2+2]

(b) A population is defined by the density function:

$$f(x;\alpha) = \frac{x^{l-1}e^{-x/\alpha}}{\Gamma(l)\alpha^l},\ 0 < x < \infty,$$

where l is a known constant. Estimate the parameter α by the method of maximum likelihood. Examine whether the estimate is consistent and unbiased. [3+2]

3. (a) The marks obtained by 18 candidates in an examination have a mean 56 and variance 65. Find 95% confidence interval for the mean of the population of marks, assuming it to be normal. Given that for 17 degrees of freedom, $P(|t| > 2.11) = 0.05$. [5]

(b) Show that approximate confidence limits for the parameter p of a binomial (n, p) population having confidence coefficient $1 - \varepsilon$ are

$$\frac{y}{n} \pm u_\varepsilon \sqrt{\frac{y(n-y)}{n^3}},$$

where y is an observed value of the population random variable X and $\pm u_\varepsilon$ are determined by

$$\int_{u_\varepsilon}^{\infty} \phi(x)dx = \frac{1}{2}\varepsilon$$

with $\phi(x)$ denoting the standard normal density function of X. [5]

4. (a) What is meant by the critical region of testing a statistical hypothesis? Define two types of errors. Also, define the level of significance and power of the test. [2+2+2]

(b) Let p be the probability that a coin will fall head in a single toss in order to test the null hypothesis $H_0 : p = 1/2$ against the alternative $H_1 : p = 3/4$. The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probabilities of two types of error. [4]

5. (a) State the Neyman-Pearson theorem for finding the best critical region in testing a statistical hypothesis. Examine whether a best critical region exists for testing the null hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1 > \theta_0$ for the parameter θ of the following distribution:

$$f(x, \theta) = \frac{1+\theta}{(x+\theta)^2}, \quad 1 \leq x < \infty.$$

[4]

(b) For a normal $(0, \sigma)$ population construct a test of the null hypothesis $H_0 : \sigma = \sigma_0$ against the alternative $H_1 : \sigma = \sigma_1 < \sigma_0$ on the basis of a sample of size n. Assume that the population mean is known. [6]