M.Sc. Examination - 2024
Semester-III
Mathematics
Course MMC-31
(Discrete Mathematics)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Notations and symbols have their usual meaning

Answer *any four* questions

1. (a) Show that

$$(A \to (B \wedge C)) \to ((B \to (D \wedge E)) \to (A \to D))$$

is a tautologous propositional form. [2]

(b) Show that

$$\begin{array}{l} (A \to B) \wedge (C \to D) \\ A \vee C \\ (A \to \sim D) \wedge (C \to \sim B) \\ \hline \therefore B \longleftrightarrow \sim D \end{array}$$

is an example of a valid argument form. [2]

(c) Examine validity of the arguments [3+3]

i)
$$\begin{array}{l} (A \vee B) \to C \\ C \to (B \vee D) \\ A \to (\sim E \to B) \\ (E \to A) \to \sim D \\ \hline \therefore B \longleftrightarrow C \end{array}$$

ii)
$$\begin{array}{l} (A \vee B) \to (C \wedge D) \\ (C \vee E) \to (\sim F \wedge G) \\ (F \vee H) \to (A \wedge I) \\ \hline \therefore \sim F \end{array}$$

2. (a) Construct a formal proof of validity for the argument [5]

$$\begin{array}{l} (\forall x)[A(x) \to \{(\exists y)(B(y) \wedge C(y)) \to D(x)\}] \\ (\exists x)\{A(x) \wedge (\forall y)(E(y) \to C(y))\} \\ \hline \therefore (\exists y)(B(y) \wedge E(y)) \to (\exists x)D(x) \end{array}$$

(b) Symbolize

i) Something attracts everything.

ii) Not all citizens are residents. [1+1]

(c) Examine validity of the argument [3]

$$\begin{array}{l} (\forall x)A(x) \to (\exists y)B(y) \\ (\forall y)B(y) \to (\exists z)C(z) \\ \hline \therefore (\exists x)A(x) \to (\forall z)C(z) \end{array}$$

3. (a) Show that every graph G with $\delta(G) \geq 2$ contains a path of length at least $\delta(G)$ and a cycle of length at least $(\delta(G)+1)$. [3]

   (b) Show that a graph G is a tree if and only if it is maximally acyclic or minimally connected. [2+2]

   (c) Show that every n vertex graph with at least n edges contains a cycle. [3]

4. (a) Define Euler graph. Show that if a connected graph is Eulerian then every vertex has even degree and the edge set can be decomposed into cycles. [1+2+2]

   (b) Define a plane graph. Establish Eulers's formula for a connected plane graph. [1+2+2]
   Show that

$$e \leq 3v - 6; \; e > 1.$$

5. (a) Among the $n-$digit quaternary sequences how many of them have an odd number of 1's ? [4]

   (b) State and prove Erdös-Szekeres theorem on the existence of large monotonic number subsequences in a finite large enough number sequence. [2+4]

6. (a) State the Inclusion-Exclusion principle. Use it to count the number of integer solutions of the system [2+3]

$$\begin{aligned} & x+y+z = 20 \\ \text{subject to } & 2 \leq x \leq 6 \\ & 4 \leq y \leq 7 \\ & -2 \leq z \leq 9. \end{aligned}$$

   (b) State and prove Burnside's theorem on counting the number of equivalence classes. [2+3]

Use separate answer script for each unit

M.Sc. Examination - 2024

Semester-III
Mathematics
Paper MMC-32
(Advanced Mathematical Statistics - I and Theory of Chaos)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Notations and symbols have their usual meaning

Unit - I : Marks - 20

(Advanced Mathematical Statistics - I)
Answer *any two* questions

1. What is a non-parametric test? Mention some advantages of using non-parametric test methods. Design the one sample runs test for randomness of a given sample. A sample of size 12 of blood glucose levels of patients who attend a hospital is given as in the following:

86 99 98 90 109 101 100 110 110 93 108 120

Use your procedure to conduct a runs test at 0.05 level of significance to see whether the sample is the result of a random process. [1+1+4+4]

2. For the normal regression analysis of the linear regression equation

$$E(Y \mid X = x) = \alpha + \beta x$$

estimate the parameters α and β. Show that the likelihood estimators $\hat{\beta}$ and $\hat{\alpha}$ are unbiased estimators of β and α. Evaluate the distributions of β and α. [2+2+3+3]

3. Consider the standard model of one-way ANOVA given by

$$Y_{ij} = \mu_i + \epsilon_{ij} \quad \text{for } i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

where $m \geq 2$ and $n \geq 2$. Assume that each random variable

$$Y_{ij} \sim N(\mu_i, \sigma^2).$$

Compute the MLE's of the parameters $\mu_i (i = 1, 2, \ldots, m)$ and σ^2. Show also that the null hypothesis

$$H_0 : \ \mu_1 = \mu_2 = \ldots = \mu_m = \mu(\text{,say})$$

is rejected whenever the test statistic $\mathcal{F}$ satisfies

$$\mathcal{F} = \frac{SS_B/(m-1)}{SS_W/(m(n-1)} > F_\alpha(m-1, m(n-1);$$

α is the significance level of the test and $F_\alpha(m-1, m(n-1)$ denote the $100(1-\alpha)$ percentile of the $F-$distribution. [4+6]

Unit - II : Marks - 20

(Theory of Chaos)

Answer *any two* questions

1. (a) What do you mean by Hopf Bifurcation?
Discuss Hopf bifurcation for the system $\dot{r} = r(\mu - r^2)$, $\dot{\theta} = -1$; $\mu \in \mathbb{R}$. [2+4]

   (b) With suitable example explain the one dimensional Transcritical Bifurcation by plotting the phase portrait and bifurcation diagram for different values of the parameter. [2]

   (c) With suitable example explain the one dimensional Saddle Node Bifurcation by plotting the phase portrait and bifurcation diagram for different values of the parameter. [2]

2. (a) Define Liapunov Exponent.Show that the Liapunov Exponent can be written in the form $\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log |f'(x_i)|$. [1+2]

   (b) Obtain the Liapunov Exponents of Tent Map, Bernoulli's shift map and Skinny Baker's map. [3]

   (c) Show that Logistic map $x_{n+1} = 4x_n(1 - x_n)$ is chaotic. [4]

3. (a) By constructing the middle third Cantor set, show that it has zero measure. [2]

   (b) Construct the Shirpinsky triangle and show that the area of the Shirpinsky triangle of the nth stage will be $\sqrt{\frac{3}{4}} \times (\frac{3}{4})^n$. [2]

   (c) Considering the iteration $z_{n+1} = z_n^2 + c$, $n = 0, 1, 2, \cdots$ in a complex plan, how Julia set and Mendelbort set are constructed. [2]

   (d) Define Box Dimension of any object. Show that the Box Dimension can be expressed as $\lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}$. [2]

   (e) Find the Similar dimension and Box dimension of Shirpinsky triangle and Von Kock Curve. [2]

M.Sc. Examination - 2024

Semester-III

Mathematics

Paper: MMC-33

(Fluid Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin
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Answer *any four* questions

1. (a) Show that equation of continuity by Lagrangian method and that by Eulerian method are equivalent without deriving them individually. [3]

   (b) Consider a two dimensional incompressible steady flow field with velocity components in spherical coordinates (r, θ, ϕ) given by

$$\begin{aligned} v_r &= c_1\left(1-\frac{3}{2}\frac{r_0}{r}+\frac{1}{2}\frac{r_0^3}{r^3}\right)\cos\theta, \\ v_\theta &= -c_1\left(1-\frac{3}{4}\frac{r_0}{r}-\frac{1}{4}\frac{r_0^3}{r^3}\right)\sin\theta, \\ v_\phi &= 0,\ r \geq r_0 > 0, \end{aligned}$$

   where c_1 and r_0 are arbitrary constants. Is the equation of continuity satisfied? [3]

   (c) If $\phi = xf(r)$ be a possible form of the velocity potential of an incompressible liquid motion then find the velocity vector. Given that the liquid speed $q \to 0$ as $r \to \infty$, deduce that the surfaces of constant speed are $(r^2+3x^2)r^{-8} = \text{constant}$. [4]

2. (a) Derive the condition that must be satisfied by a boundary surface of a fluid in motion. [3]

   (b) Determine the restriction/s on f_1, f_2, f_3 if

$$(x^2/a^2)f_1(t) + (y^2/b^2)f_2(t) + (z^2/c^2)f_3(t) = 5$$

   be a possible boundary surface of a liquid in motion. [3]

   (c) Show that if the velocity potential of an irrotational fluid motion is equal to $A(x^2+y^2+z^2)^{-\frac{3}{2}}z\tan^{-1}(y/x)$, the lines of flow will lie on the series of the surfaces

$$x^2+y^2+z^2 = c^{\frac{2}{3}}(x^2+y^2)^{\frac{2}{3}}.$$

   [4]

3. (a) Show that if the motion of an ideal fluid, for which density is a function of pressure only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the streamlines and vortex lines. [3]

   (b) What is the physical significance of stream function? Give reason in support of your answer. [3]

(c) In an irrotational motion in two dimensions, prove that

$$(\partial q/\partial x)^2 + (\partial q/\partial y)^2 = q\nabla^2 q.$$

[4]

4. (a) State and prove the Milne-Thomson circle theorem. [1+3]

(b) Derive complex potential due to a doublet in two-dimensions. [3]

(c) Let there be a source of strength m at $(a, 0)$ and a sink $-m$ at $(0, a)$. Find ϕ, ψ, w and velocity q. [3]

5. (a) State and prove the Kelvin's minimum energy theorem. [1+4]

(b) Show that if Σ be the boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, then the mean value of ϕ on Σ is of the form

$$\overline{\phi} = M/R + C,$$

where M, C are constants, provided that the fluid extends to infinity and is at rest there. [3]

(c) Calculate the circulation round a unit circle with centre at the origin for a velocity field

$$\vec{q} = (\hat{j}x - \hat{i}y)/(x^2 + y^2).$$

[2]

6. (a) Determine the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity, with velocity U in the direction of $x-$ axis. Hence deduce the motion relative to a fixed cylinder and show that maximum value of velocity on the cylinder is twice the velocity of free stream. [5]

(b) Discuss the generalized Coutte flow. Determine the volumetric flow rate, shear stress and coefficient of friction, for the said flow. [5]

Use separate answer script for each unit

M.Sc. Examination, 2024

Semester-III
Mathematics
Paper: MMC-34
(Calculus of Variations and Special Functions)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
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Unit-I (Full Marks: 20)
(Calculus of Variations)
Answer ***any two*** questions.

1. (a) Show that the necessary conditions for the existence of extremals of the functional

$$I[x(t), y(t)] = \int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})dt,$$

subject to given fixed boundary conditions at t_1 and t_2 are $F_x - \frac{d}{dt}(F_{\dot{x}}) = 0$ and $F_y - \frac{d}{dt}(F_{\dot{y}}) = 0$. [5]

(b) A particle moves on a smooth curve, joining two fixed points A and B, under gravity, starting from rest at A, find the equation of path in order that the time from A to B is minimum. [5]

2. (a) Show that the Euler-Ostrogradsky equation for the functional

$$S[z(x, y)] = \iint_{\Omega} \left\{\sqrt{1 + z_x^2 + z_y^2}\right\} dxdy,$$

where the values of z are prescribed on the boundary $\partial\Omega$ of the domain Ω, is given by $z_{xx}(1 + z_y^2) - 2z_x z_y z_{xy} + z_{yy}(1 + z_x^2) = 0$. [5]

(b) Find the shortest distance between the point $y^2 = 4x$ and the straight line $x + y + 5 = 0$. [5]

3. (a) Find the extremal of the functional $I[y(x)] = \int_0^{\pi} (y'^2 - y^2)dx$ with $y(0) = 0$, $y(\pi) = 1$ subject to the condition $\int_0^{\pi} ydx = 1$. [5]

(b) Using Legendre condition, test for an extremum of the functional

$V[y(x)] = \int_1^2 \frac{x^3}{y'^2} dx$, $y(1) = 1$, $y(2) = 4$. [5]

Unit-II (Full Marks: 20)
(Special Functions)
Answer ***any two*** questions.

1. (a) Using the generating function for Legendre polynomials $P_n(x)$ of degree n prove the following recurrence relation
$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$. [3]

(b) Prove that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$, where δ_{mn} denotes Kronecker delta. [4]

(c) Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomial. [3]

2. (a) Solving Bessel's equation in series for $n = 0$, find Bessel's function of zeroth order. [4]

(b) Prove the following recurrence relations for Bessel's function $(J_n(x))$
(*i*) $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$,
(*ii*) $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$,
(*iii*) $J_n'(x) = J_{n-1}(x) - (n/x)J_n$. [2+2+2]

3. (a) Show that $G(x, t) = \frac{1}{1-t} e^{-\frac{xt}{1-t}}$ is a generating function for Laguerre polynomial. Hence show that
$(n + 1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x)$ and $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$. [3+2+2]

(b) Show that $H_n'(x) = 2nH_{n-1}(x)$ where $H_n(x)$ is the Hermite polynomial of degree n. [3]

Use separate answer script for each unit

M. Sc. Examination-2024

Semester-III
Mathematics
Elective Course: MMC-35
(Galois Theory -I and Multivariable Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Galois Theory I (Full Marks: 20)

Answer *any four* questions

1. (a) Prove that $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a field.
   (b) Define a prime field and give two examples of fields that are not prime.
   (c) Prove that any prime field is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime number p. [2+1+2]

2. (a) Prove that $X^2 - 7$ is irreducible over $\mathbb{Q}(\sqrt{3})$. Hence find $[\mathbb{Q}(\sqrt{3}, \sqrt{7}) : \mathbb{Q}]$ and determine a $\mathbb{Q}$-basis of $\mathbb{Q}(\sqrt{3}, \sqrt{7})$.
   (b) Determine, with proof, an element $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{7}) = \mathbb{Q}(\alpha)$. [3+2]

3. (a) Prove that there exists a field homomorphism $\phi : \mathbb{Q}(\pi) \to \mathbb{Q}(e)$ such that $\phi(\pi) = e$.
   (b) Let K, L and F be fields such that $K \subseteq L \subseteq F$. Assume that F is an algebraic extension of L and L is an algebraic extension of K. Prove that F is an algebraic extension of K. [2+3]

4. (a) Let $f(X) \in K[X]$ be a polynomial of degree $n \geq 1$ and let F be a splitting field of f over K. Prove that $[F : K] \leq n!$.
   (b) Can $\mathbb{C}$ be the splitting field of some $f(X) \in \mathbb{Q}[X]$? Justify your answer. [3+2]

5. (a) Let K be a field. Prove that there exists a field $F \supseteq K$ such that F is an algebraic extension of K and F is algebraically closed.
   (b) Let K be a perfect field with $\text{char}(K) = p > 0$. Prove that $K = K^p$. [2+2+1]

6. (a) Let F be a field such that F^* is a cyclic group. Prove that F is a finite field.
   (b) Let K be a finite field with q elements. If q is odd, then prove that exactly $\frac{q-1}{2}$ elements of K^* are squares in K^*. [3+2]

Unit-II : Multivariable Analysis (Full Marks: 20)

Answer any two questions

1. (a) Prove that a linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is equal to its own derivative at every point. [2]
   (b) If f is a C^1 function then prove that f is differentiable. [4]
   (c) If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are continuous functions then show that $g \circ f$ is also continuous. [2]
   (d) Prove that the product of two differentiable functions f and g on $\mathbb{R}^n$ is differentiable. [2]

2. (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function. Prove that $D_i D_j f = D_j D_i f$. [3]
   (b) Define directional derivative $D_v f(a)$ of the function $f : \mathbb{R}^n \to \mathbb{R}^m$ in the direction v. Show that $D_v f(a) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(a)$, where $v = (v_1, v_2, ..., v_n)$. [2]
   (c) State and prove Taylor's theorem for a C^{k+1} function $f : \mathbb{R}^n \to \mathbb{R}$. [4]
   (d) Find the kth degree Taylor's polynomial of $e^{x_1 + ... + x_n}$. [1]

3. (a) Define the quadratic form $q(h)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ at a critical point a. Show that f has neither a maximum nor a minimum at the point a if $q(h)$ is non definite. [4]
   (b) State and prove the inverse mapping theorem. [4]
   (c) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function and $det(f'(a)) \neq 0$. Prove that there is $\delta > 0$ such that f is one-to-one in the ball $B(a, \delta)$ [2]

M. Sc. Examination, 2024

Semester-III

Mathematics

Paper : MMO-31 (A9)

(Mathematical Pharmacology-I)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
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Answer ***any four*** questions.

1. Define biphasic absorption of drug. Using rate theory, prove that the equilibrium response $\gamma = \phi \frac{k_2 x}{x + \frac{k_2}{k_1}}$, k_1 and k_2 are the association and dissociation rate constants, respectively; x is the concentration of drug applied and ϕ is a constant. [3+7]

2. Using conservation laws for receptor and ligand, formulate the governing equation for receptor/ligand complex ($c(t)$) for simple monovalent cell surface binding model. Discuss the case for constant ligand concentration. [3+7]

3. Define specific and non-specific binding of drug. Assuming interconversion of receptor and complex states, and equilibrium binding of ligand to both receptor forms, obtain the transient solution for both complexes. [3+7]

4. Define endocytosis. Formulate a single endosome model in which monovalent ligands bind to monovalent receptors. Write down the descriptions of variables and parameters involved. Nondimensionalise the equations. [3+3+4]

5. Using the theory of random walk, derive Fick's first law. Formulate model equation with appropriate initial and boundary conditions when a drug is initially confined to a linear band of width $2w$ and concentration c_0. Solve the equation. [4+3+3]

6. Formulate a problem of one-dimensional drug transport in a semi-infinite domain where the volumetric rate of elimination is first-order. Using proper intitial and boundary conditions, solve the equation. Hence obtain the steady-state solution. [3+5+2]