M. Sc. Examination-2024

Semester-IV

Mathematics

Course: MMC-41(New)

(Differential Geometry and Manifold Theory)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. Define the Frenet frame for a curve Γ in $\mathbb{R}^3$. Establish the Serret-Frenet formula for the Frenet frame. For a regular curve $\vec{r}(t)$, derive expressions for curvature κ and the torsion τ in terms of derivatives of $\vec{r}(t)$. State fundamental theorem of curves. Hence or otherwise, show that a curve with $\tau = 0$ is a plane curve. [2+3+2+1+2]

2. Define tangent vector at a point $p \in \mathbb{R}^n$. What do you mean by tangent space $T_p\mathbb{R}^n$ at a point $p \in \mathbb{R}^n$, tangent bundle $T\mathbb{R}^n$ of $\mathbb{R}^n$. Derive a basis for $T_p\mathbb{R}^n$ in terms of the local coordinates $x^1, x^2, \ldots, x^n$ of $\mathbb{R}^n$. Define vector fields on $\mathbb{R}^n$ and its action on smooth functions $C^\infty(U)$, where U is some neighborhood of $\mathbb{R}^n$. Prove that each vector field $\mathbf{X} \in \mathcal{X}(U)$ is a linear derivation on $C^\infty(U)$. [1+2+2+1+4]

3. Define Hodge $*$-map between the space of forms. Determine the action of the Hodge map on the differential form $dx^2 \wedge dx^3$ in $\mathbb{R}^4$. Derive expressions for grad($\vec{\nabla}$), div($\vec{\nabla}\cdot$), curl($\vec{\nabla}\times$) and Laplacian (∇^2 or $\triangle$) operators as the combination of d operator and $*$-map in the spaces associated with $\mathbb{R}^3$. Hence or otherwise, derive expressions for Laplacian (∇^2 or $\triangle$) operator in an arbitrary curvilinear coordinate in $\mathbb{R}^3$. [1+2+4+3]

4. Introduce the inner product in the tangent space of $\mathbb{R}^n$. Define metric tensor with the aid of the inner product just introduced. Prove that g_{ij} may be regarded as mapping $g : T\mathbb{R}^n \times T\mathbb{R}^n \to \mathbb{R}$ and $g : T^*\mathbb{R}^n \to T\mathbb{R}^n$. Derive expressions for g_{ij} in the surface of revolution $\mathbf{x}(u^1, u^2) = (f(u^1)\cos u^2, f(u^1)\sin u^2, g(u^1)) \in \mathbb{R}^3$. [2+2+3+3]

5. Derive expressions for second partial derivatives of $\mathbf{x}(u^1, u^2)$ with respect to the parameters u^1, u^2 in terms of Christoffel symbols of the second kind. Establish the formula $\Gamma_{ijk} = \mathbf{x}_{ij} \cdot \mathbf{x}_k = \frac{1}{2}(g_{ik;j} + g_{jk;i} - g_{ij;k})$. Prove that in case of an orthogonal curvilinear coordinate system (u^1, u^2) on a surface in $\mathbb{R}^3$ with first fundamental form $ds^2 = E(du^1)^2 + G(du^2)^2$, $\Gamma^1_{11} = \frac{E_{u^1}}{2E}$, $\Gamma^2_{22} = \frac{G_{u^2}}{2G}$, $\Gamma^1_{12} = \Gamma^1_{21} = \frac{E_{u^2}}{2E}$, $\Gamma^2_{21} = \Gamma^2_{12} = \frac{G_{u^1}}{2G}$, $\Gamma^1_{22} = -\frac{G_{u^1}}{2E}$ and $\Gamma^2_{11} = -\frac{E_{u^2}}{2G}$. [4+6]

6. State the definition of covariant derivative $\vec{\nabla}_{\mathbf{X}}(\mathbf{Y})$. Introduce Riemann-Christoffel curvature tensor R^l_{ijk}. When a Riemannian space is said to be flat? Hence or otherwise, establish conditions on the coefficients $F_i(x, y)$, $i = 0, 1, 2, 3$ of the second-order nonlinear ordinary differential equation $y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F_0(x, y) = 0$ by its projection onto $(\dot{x}(t), \dot{y}(t))$-space so that the above second-order nonlinear can be reducible to the linear equation $z'' = 0$ by a suitable point transformation $(x, y(x)) \to (s, z(s))$. [1+2+1+6]

Internal Assesment (I)-2024
Semester-IV
Mathematics
MMC-41(New)
(Differential Geometry and Manifold Theory)

Time: 45 Mins

Full Marks: 06

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **any two** questions.
Submit answer of all questions within one week from this internal assessment.

1. Define tangent vector at a point $p \in \mathbb{R}^n$. What do you mean by tangent space $T_p\mathbb{R}^n$ at a point $p \in \mathbb{R}^n$, tangent bundle $T\mathbb{R}^n$ of $\mathbb{R}^n$. Derive a basis for $T_p\mathbb{R}^n$ in terms of the local coordinates $x^1, x^2, \ldots, x^n$ of $\mathbb{R}^n$. Define vector fields on $\mathbb{R}^n$ and its action on smooth functions $C^\infty(U)$, where U is some neighborhood of $\mathbb{R}^n$. Prove that each vector field $\mathbf{X} \in \mathcal{X}(U)$ is a linear derivation on $C^\infty(U)$. [3]

2. Introduce differential forms and wedge products on the manifold $\mathbb{R}^n$. What do you mean by the exterior derivative of differential forms belonging in $\Lambda_p(\mathbb{R}^n)$? Prove the formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \bigwedge^p$. When a form is said to be exact. Prove that every exact form is closed in $\mathbb{R}^n$. Is the converse true? Prove it if yes. [3]

3. How do you introduce the inner product in the tangent space? Define metric tensor with the aid of the inner product just introduced. Prove that g_{ij} may be regarded as mapping $g : T\mathbb{R}^n \times T\mathbb{R}^n \to \mathbb{R}$ and $g : T^*\mathbb{R}^n \to T\mathbb{R}^n$. Derive expressions for g_{ij} in the toroidal coordinate system $((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$ and verify whether it is orthogonal? [3]

4. Define Hodge $*$ map between the space of forms. Determine the action of the Hodge map on the differential forms dx^2 and $dx^1 \wedge dx^3$ in $\mathbb{R}^4$. Derive expressions for grad($\vec{\nabla}$), div($\vec{\nabla}\cdot$), curl($\vec{\nabla}\times$) and Laplacian (∇^2 or $\triangle$) operators as the combination of d operator and $*$-map in the spaces associated with $\mathbb{R}^3$. [3]

5. Derive expressions for grad($\vec{\nabla}$), div($\vec{\nabla}\cdot$), curl($\vec{\nabla}\times$) and Laplacian (∇^2 or $\triangle$) operators in spherical polar coordinates. [3]

6. State and prove Cartan's first and second structure equations. Find expressions for frame and matrix-valued connection 1-form in spherical polar coordinates. [3]

7. Define the Frenet frame for a curve in $\mathbb{R}^3$. Establish the Serret-Frenet formula for the Frenet frame. For a regular curve $\vec{r}(t)$, derive expressions for curvature κ and the torsion τ in terms of derivatives of $\vec{r}(t)$. State fundamental theorem of curves. Explain the importance of each term in this theorem. Hence or otherwise, show that a curve with $\tau = 0$ is a plane curve. [3]

Internal Assesment (II)-2024
Semester-IV
Mathematics
MMC-41(New)
(Differential Geometry and Manifold Theory)

Time: 25 Mins

Full Marks: 06

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer **any one** questions.
Submit answer of all questions within one week from this internal assessment.

1. Define covariant derivative of a vector field $\mathbf{Y}$ in the direction of a vector field $\mathbf{X}$. Find $\bar{\nabla}_{\mathbf{X}}\mathbf{Y}$ for $\mathbf{X} = y\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$ and $\mathbf{Y} = y^2\frac{\partial}{\partial x} + x^2y\frac{\partial}{\partial y}$. Write down the covariant derivatives of a frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the direction of a vector field $\mathbf{X}$ and interpret the coefficients involve there. Define Christoffel symbols and find $\bar{\nabla}_{\mathbf{e}_i}\mathbf{X}$ for $\mathbf{X} = v^k\mathbf{e}_k$. [6]

2. Define the first and the second fundamental form on a surface in $\mathbb{R}^3$. Describe the length of a vector field $\mathbf{X}$ and the angle between two vector fields $\mathbf{X}$ and $\mathbf{Y}$ involving the metric tensors appearing in the first fundamental form of a surface in $\mathbb{R}^3$. Find the expression for the first and the second fundamental forms on a surface of revolution $\mathbf{x}(r, \theta) = (r\cos\theta, r\sin\theta, f(r))$. [6]

3. Define the normal curvature of a surface in $\mathbb{R}^3$ in the direction of a unit vector field $\mathbf{v} = v^k\mathbf{e}_k$. Hence or otherwise, introduce principal curvatures, principal directions, and the Gauss curvature at a point P on a surface in $\mathbb{R}^3$. Establish that the Gauss curvature at a point P on a surface in $\mathbb{R}^3$ may be obtained by using the formula $\frac{L}{g}$. Interpret this result. Find the Gauss curvature for the surface $\mathbf{x}(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$. [6]

4. Define Gauss mapping of some smooth or regular neighborhood Ω of a point $\mathrm{P}(\mathbf{x}((\mathbf{u}_0^1, \mathbf{u}_0^2)))$ on a surface in $\mathbb{R}^3$. Prove that the spherical normal image of a torus $((R + r\cos u^1)\cos u^2, (R + r\cos u^1)\sin u^2, r\sin u^1)$; R and r being constants, is the whole splere S^2. Establish the formula $\hat{U}_1 \times \hat{U}_1 = K(\mathbf{x}_1 \times \mathbf{x}_2)$. Interpret the constant K. State the equations of Weingarten. [6]

5. Derive expressions for second derivatives of $\mathbf{x}(u^1, u^2)$ with respect to the parameters u^1, u^2 in terms of Christoffel symbols of the second kind. Establish the formula $\Gamma_{ijk} = \mathbf{x}_{ij} \cdot \mathbf{x}_k = \frac{1}{2}(g_{ik;j} + g_{jk;i} - g_{ij;k})$. Prove that in case of an orthogonal curvilinear coordinate system (u^1, u^2) on a surface in $\mathbb{R}^3$ with first fundamental form $ds^2 = E(du^1)^2 + G(du^2)^2$, $\Gamma^1_{11} = \frac{E_{u^1}}{2E}$, $\Gamma^2_{22} = \frac{G_{u^2}}{2G}$, $\Gamma^1_{12} = \Gamma^1_{21} = \frac{E_{u^2}}{2E}$, $\Gamma^2_{21} = \Gamma^2_{12} = \frac{G_{u^1}}{2G}$, $\Gamma^1_{22} = -\frac{G_{u^1}}{2E}$ and $\Gamma^2_{11} = -\frac{E_{u^2}}{2G}$. [6]

6. Derive the equation for geodesic on a surface in $\mathbb{R}^3$. Find the equation for geodesic on the unit sphere S^2 and the surface of revolution $\mathbf{x}(u^1, u^2) = (\sinh u^1\cos u^2, \sinh u^1\sin u^2, 5\cosh u^1)$, $u^1 \neq 0$ in $\mathbb{R}^3$. [6]

7. Define the Riemann-Christoffel curvature and the Ricci tensor. Establish that both tensors reflect the intrinsic property of a surface in $\mathbb{R}^3$. Prove that Riemann-Christoffel curvature is antisymmetric concerning some indices that you must determine. Prove the Gauss theorem Egregium, i.e., Gauss curvature of a surface in $\mathbb{R}^3$ is an intrinsic property. [6]

$R^{\gamma}_{\alpha\beta\delta} = b_{\alpha\delta}b^{\gamma}_{\beta} - b_{\alpha\beta}b^{\gamma}_{\delta}$

8. Establish conditions on the coefficients $F_i(x,y)$, $i=0,1,2,3$ of the second-order nonlinear ordinary differential equation $y''+F_3(x,y)y'^3+F_2(x,y)y'^2+F_1(x,y)y'+F_0(x,y)=0$ by its projection onto $(\dot{x}(t),\dot{y}(t))$-space so that the above second-order nonlinear can be reducible to the linear equation $z''=0$ by a suitable point transformation $(x,y(x))\to(s,z(s))$. [6]

9. Define Chart, atlas, maximal atlas, and manifold structure on a topological space $\mathcal{M}$. Provide a manifold structure on the topological space $S^n=\{(x^0,x^1,\cdots,x^n):(x^0)^2+(x^1)^2+\cdots+(x^n)^2=1\}$. [6]

10. Define Lie group. Prove that $S=\left\{\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix};\ a,b,c\in\mathbb{R}\right\}$ is a Lie group for an appropriate algebraic operation on matrices to be determined by you. Give the explicit form of functions appearing in the binary compositions of any two elements and the inversion of any element in S. [6]

11. Define Lie group of point transformations on a geometric space $\mathcal{G}$. Prove that the set of transformations $T_\epsilon:\mathbb{R}^2(\equiv\mathcal{G})\to\mathcal{G}$ defined by $\mathcal{G}\ni(x,y)\to(x^*,y^*)=T_\epsilon(x,y)=(x\cos\epsilon-y\sin\epsilon,x\sin\epsilon+y\cos\epsilon),\epsilon\in S=\mathbb{R}\ mod\ 2\pi$ is a Lie group of point transformations. Give the explicit form of functions appearing in the binary compositions of any two elements and the inversion of any element in S. Write down the Lie group of transformations mentioned above in its infinitesimal form and derive the expression for its infinitesimal generator. [6]

Use separate answer script for each unit

M.Sc. Examination-2024

Semester-IV
Mathematics
MME-41
(Galois Theory -II and Algebraic Topology)

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Galois Theory II (Full Marks: 20)

Answer ***any four*** questions.

1. (a) Let $f(x) \in K[x]$ be a nonconstant polynomial. Show that there exists an extension F of K such that F contains a root of $f(x)$.

   (b) Find a splitting field S of $x^3 - 2$ over $\mathbb{Q}$ and $[S : \mathbb{Q}]$. [3+2]

2. (a) Give two examples of finite field extensions, with justification, one of which is a normal extension and the other is not.

   (b) Let $f(x) \in K[x]$ be a non-constant polynomial. If S_1 and S_2 are two splitting fields of $f(x)$ over K, prove that $G(S_1/K) \simeq G(S_2/K)$. [3+2]

3. (a) If F/K be a finite extension, show that $|G(F/K)| \leqslant [F : K]$.

   (b) Show that $G(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}_2$. [3+2]

4. (a) If F/K is a finite normal and separable extension, show that F is a splitting field of a separable polynomial $f(x) \in K[x]$.

   (b) Let F/K be a finite extension and H be a subgroup of $G(F/K)$. Show that $F_H = F_{G(F/F_H)}$. [3+2]

5. (a) If S is a splitting field of $x^3 - 2$ over $\mathbb{Q}$, then show that $G(S/\mathbb{Q}) \simeq S_3$.

   (b) Use the Fundamental Theorem of Galois theory to prove that there is exactly one intermediate field L of $S/\mathbb{Q}$ such that $L/\mathbb{Q}$ is a normal extension. [3+2]

6. (a) Define radical extension and solvability of a polynomial by radicals.

   (b) Show that there is a quintic which is not solvable by radicals. [2+3]

Unit-II : Algebraic Topology (Full Marks: 20)

Answer **any four** questions

1. (a) Let X be a connected and locally path-connected topological space. Prove that X is path-connected. [3]

   (b) Prove that $[0,1]$ is not homeomorphic to S^1. [2]

2. (a) Let $f_1, g_1 : X \to Y$ and $f_2, g_2 : Y \to Z$ be continuous functions such that f_1 is homotopic to g_1 and f_2 is homotopic to g_2. Prove that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$. [3]

   (b) When is a topological space called contractible? Prove that $[0,1]$ is contractible. [2]

3. (a) Prove that the relation "is of same homotopy type" is an equivalence relation on the set of all topological spaces. [2]

   (b) Prove that a topological space X is contractible if and only if X is of the same homotopy type as that of a singleton space $P = \{p\}$. [3]

4. (a) Prove that a contractible topological space is path-connected. [3]

   (b) Let X and Y be topological spaces with X being connected and Y being discrete. Prove that if f and g are two continuous homotopic maps, then $f = g$. [2]

5. (a) Prove that a retract of a contractible space is contractible. [3]

   (b) If $r : X \to A$ is a retraction map, then prove that the induced map $r_\#$ is surjective. [2]

6. (a) If X and Y are path-connected topological spaces of same homotopy type and let $x_0 \in X$ and $y_0 \in Y$. Prove that the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic. [3]

   (b) If X is a simply connected topological space, prove that any two paths having the same end points are homotopic. [2]

Internal of MME-41

- Answer both the questions; total marks is 6. Both the questions carry equal marks.

(1) Prove that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path-connected.

(2) Let X and Y be topological spaces and let $C(X, Y)$ be the set of all continuous maps from X to Y. Prove that "being homotopic to" is an equivalence relation on $C(X, Y)$.

M. Sc. (Sem - IV) First Internal Examination
Galois Theory

Time: 30 mins. **Full Marks: 5**

Answer any one. $1 \times 5 = 5$

1. Let K be a field and $f(x) \in K[x]$ be a nonconstant polynomial. Show that there exists a splitting field S of $f(x)$ over K.

   Find a splitting field S of $x^3 - 2$ over $\mathbb{Q}$. Also find $[S : \mathbb{Q}]$.

2. Let K be a field and $f(x) \in K[x]$ be a nonconstant polynomial. Prove that any two splitting fields of $f(x)$ over K are isomorphic.

   Find a splitting field S of $x^2 + 2x + 2$ over $\mathbb{Q}$. Also find $[S : \mathbb{Q}]$.

M. Sc. Examination-2024

Semester-IV

Mathematics

Paper: MMC 42

(Operations Research)

Time: 3 Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) How can you detect that the LPP has a basic non-degenerate feasible solution? [2]

   (b) Introducing slack, surplus and artificial variables, write down the following LPP in the standard form and hence solve by revised simplex method (RSM):

$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= 4x_1 + 3x_2 \\ \text{subject to } 2x_1 + x_2 &\geq 10, \\ -3x_1 + 2x_2 &\leq 6, \\ x_1 + x_2 &\geq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

   Is the optimal solution unique? Justify your answer. [7+1]

2. (a) What are the main objectives of sensitivity analysis in operations research? [2]

   (b) Consider the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= x_1 + 3x_2 \\ \text{subject to } x_1 &\leq 5, \\ x_1 + 2x_2 &\leq 10, \\ x_2 &\leq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

   (a) Find the range of the objective function coefficients c_1 of the variable x_1 and c_2 of the variable x_2 such that the optimality is unaffected.

   (b) Discuss the effect of change of b from $[5, 10, 4]$ to $[10, 15, 8]$. [4+4]

3. (a) What is the difference between pure and mixed-type integer linear programming problem (ILPP)? [2]

   (b) Find the optimum integer solution of the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 7x_1 + 9x_2 \\ \text{subject to } -x_1 + 3x_2 &\leq 6, \\ 7x_1 + x_2 &\leq 35, \\ x_1,\ x_2 &\geq 0 \text{ and are integers.} \end{aligned}$$

   [8]

4. (a) Use the Kuhn-Tucker necessary and sufficient conditions to solve the following nonlinear programming problem (NLPP):

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 2x_1 - x_1^2 + x_2 \\ \text{subject to } 2x_1 + 3x_2 &\leq 6, \\ 2x_1 + x_2 &\leq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

   [5]

   (b) Solve the following NLPP graphically:

$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= (x_1 - 1)^2 + (x_2 - 2)^2 \\ \text{subject to } 0 &\leq x_1 \leq 2, \\ 0 &\leq x_2 \leq 1. \end{aligned}$$

   [5]

5. (a) Obtain mathematical expressions for the optimal lot size and corresponding annual cost in an Economic Order Quantity (EOQ) manufacturing model without shortages. [6]

(b) An automobile factory manufactures a particular type of gear within the factory. This gear is used in the final assembly. The particulars of this gear are: demand rate $r = 42000$ units per year, production rate $k = 105000$ units per year, set-up cost $C_s =$ Rs. 1500 per set-up and carrying cost $C_1 =$ Rs. 45 per unit per year. Find the economic batch quantity (EBQ) and cycle time. 2160, 19, 19 [2+2]

6. (a) For the queuing model $M/M/1 : \infty/FCFS$, having Poisson arrival, Poisson departure (exponential service time), single server, infinite capacity and queue discipline is first come first service; find the average number of customers in the system at any time. [6]

(b) In a supermarket, the average arrival rate of customer is 9 in every 5 minutes, following Poisson process. The average time taken by the cashier to list and calculate the customer's purchase is 2.7 minutes, following exponential distribution.

(i) What is the probablity that the queue length exceeds six?

(ii) Find L_s, L_q, W_s and W_q with the help of Little's formula. [2+2]

M. Sc. Semester-IV Examination-2024 (Internal)

Mathematics

Paper: MMC 42 (Operations Research)

Time: 1.5 Hours **Full Marks: 20**

Answer ***all*** the questions.

Internal-I (Answer any one question)

1. Solve the following linear programming problem (LPP) by revised simplex method (RSM):

$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= 12x_1 + 20x_2 \\ \text{subject to } 6x_1 + 8x_2 &\geq 100, \\ 7x_1 + 12x_2 &\geq 120, \\ x_1, x_2 &\geq 0. \end{aligned}$$

[10]

2. Use revised simplex method (RSM) to solve the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 4x_1 + 8x_2 + 3x_3 \\ \text{subject to } x_1 + x_2 &\geq 2, \\ 2x_1 + x_3 &\geq 5, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[10]

Internal-II (Answer any one question)

3. Consider the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= x_1 + 3x_2 \\ \text{subject to } x_1 &\leq 5, \\ x_1 + 2x_2 &\leq 10, \\ x_2 &\leq 4, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

(a) Solve the LPP.
(b) Find the range of the objective function coefficient c_2 of the variable x_2 such that the optimality is unaffected.
(c) Discuss the effect of change of b from $[5, 10, 4]$ to $[10, 15, 8]$. [4+3+3]

4. Consider the following LPP:

$$\begin{aligned} \text{Maximize } f(x_1, x_2) &= 2x_1 + 3x_2 \\ \text{subject to } x_1 + x_2 &\leq 8, \\ x_1 + 2x_2 &= 5, \\ 2x_1 + x_2 &\leq 8, \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

(a) Solve the LPP.
(b) Check whether the optimality is affected if the cost coefficients are changed from $[2, 3]$ to $[4, 5]$.
(c) Comment on the optimal solution if the new constraint $5x_1 + 7x_2 \leq 23$ is incorporated into the optimization problem. [3+4+3]

M. Sc. Examination-2024

Semester-IV

Mathematics

MMO 41 : Optional Paper

P 9 : Rings and Modules-II

Time: Three Hours **Full Marks: 40**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
Every ring R contains unity and every module is unitary.

Answer ***any four*** questions.

1. (a) Let $J(R)$ be the intersection of all maximal left ideals of R. Show that $J(R) = \{a \in R \mid 1 - ras$ is invertible for every $r, s \in R\}$ [4]
   (b) Let F be a field. Show that $M_n(F)$ is a Jacobson semisimple ring. [3]
   (c) Show that every Jacobson semisimple ring is a subdirect product of primitive rings. [3]

2. (a) Show that an abelian group is an injective $\mathbb{Z}$-module if and only if it is divisible. [3]
   (b) Prove that an R-module M is injective if and only if for every left ideal L of R and every R-morphism $f : L \longrightarrow M$, there exists an R-morphism $h : R \longrightarrow M$ that extends f. [4]
   (c) Prove that an R-module I is injective if and only if for every monomorphism $\alpha : I \longrightarrow M$, $Im\ \alpha$ is a direct summand of M. [3]

3. (a) Let M be an R-module. If M is a sum of a family of simple modules, show that every submodule of M is a direct summand of M. [4]
   (b) Let L, L' be two left ideals of a ring R such that $LL' \neq 0$. Show that $L \simeq L'$ as left R-modules. [3]
   (c) Let R be a left semisimple ring. Show that every simple component of R is a simple subring of R. [3]

4. (a) State and prove a necessary and sufficient condition for a ring R to be a subdirect product of a family of rings $\{R_\alpha \mid \alpha \in \Delta\}$. [4]
   (b) Prove that every ring is a subdirect product of subdirectly irreducible rings. [3]
   (c) Show that every Boolean ring is a subdirect product of isomorphic copies of $\mathbb{Z}_2$. [3]

5. (a) Let M_R and ${}_RN$ be two R-modules. Show that any two tensor products of M and N are isomorphic. [3]
   (b) Let M be a right R-module and $\{N_\alpha \mid \alpha \in \Delta\}$ be a family of left R-modules. Prove that $M \otimes (\oplus N_\alpha) \simeq \oplus(M \otimes N_\alpha)$. [4]
   (c) If $d = \gcd(m, n)$, show that $\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_d$. [3]

6. (a) Give an example of an injective module. [2]
   (b) Find $J(\mathbb{Z}_{100})$. [2]
   (c) Show that every vector space is semisimple. [2]
   (d) Show that $\mathbb{Z}$ is not a semisimple $\mathbb{Z}$-module. [2]
   (e) Show that $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q} = 0$. [2]