Four Year Undergraduate Examination - 2025

Semester-II (NEP)

Mathematics

MJMA 3 (Algebra III)

Time: ~~Four~~ Three Hours

Full Marks: 80

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Group-A (Group Theory)

Answer **any four** questions. $(4 \times 10 = 40)$

1. (a) Let H be subgroup of a group G, and $a, b \in G$. Show that $aH = bH$ if $a^{-1}b \in H$. [3]
   (b) Let H and K be two subgroups of a group G, and $a, b \in G$. If $aH = bK$, then show that $H = K$. [3]
   (c) Let H and K be two finite subgroups of a group G. Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$ [4]

2. (a) State and prove the Lagrange's theorem on finite groups. [4]
   (b) Let G be a group of order 91. Show that every proper subgroup of G is cyclic. [3]
   (c) Show that every finite subgroup of the group $\mathbb{C}^*$ of all nonzero complex numbers is cyclic. [3]

3. (a) Let G and H be two finite cyclic groups. Prove that $G \times H$ is cyclic if and only if $\gcd(|G|, |H|) = 1$. [4]
   (b) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic. [3]
   (c) Find all subgroups of order 5 of the group $\mathbb{Z}_5 \times \mathbb{Z}_5$ [3]

4. (a) Let H be a subgroup of a group G. Show that H is normal in G if and only if $aHa^{-1} = H$ for all $a \in G$. [4]
   (b) Let G be a non-abelian group, and $Z(G)$ be its center. Show that the order of the quotient group $G/Z(G)$ can not be 17. [3]
   (c) Give an example of a group G that contains an element of order n for each $n \in \mathbb{N}$. [3]

5. (a) Let $f : G \longrightarrow G'$ be a group homomorphism. If K is a normal subgroup of G', then show that $f^{-1}(K)$ is a normal subgroup of G. Give an example to show that homomorphic image of a normal subgroup may not be normal. [4]
   (b) Show that $\mathbb{R} \simeq \mathbb{R}^+$ but $\mathbb{R} \not\simeq \mathbb{R}^*$. [3]
   (c) Show that the group $G = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}$. [3]

6. (a) State and prove that first isomorphism theorem for groups. [4]
   (b) Show that $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \simeq \mathbb{R}^*$. [3]
   (c) Let $f : G \longrightarrow G'$ be a group homomorphism. If $|\ker f| = 6$, then show that f is a 6-to-1 mapping. [3]

Group-B (Ring Theory)

Answer **any four** questions. $(4 \times 10 = 40)$

7. (a) Let R be a ring with 1 and $U(R)$ be the set of all units in R. Prove that $(U(R), \cdot)$ is a group. [3]
   (b) Determine $U(\mathbb{Z}[i])$. [3]
   (c) Let S be a nonempty subset of a ring R. State and prove a necessary and sufficient condition for S to be a subring of R. [4]

8. (a) Let R be a finite commutative ring with unity. Show that every nonzero element of R is a unit or a zero-divisor. [4]
   (b) Let a be a unit and b be a nilpotent element of a commutative ring R with 1. Show that $a + b$ is a unit in R. [3]
   (c) Show that S is a subring of $\mathbb{Z}$ if and only if $(S, +)$ is a subgroup of $(\mathbb{Z}, +)$. [3]

9. (a) Let X be a nonempty set. Show that the characteristic of the ring $P(X)$ is 2. Find the unity, and characterize all the units in $P(X)$. [4]
   (b) Give an example with justification of a ring R of characteristic 2, that is not a Boolean ring. [3]

(c) Find all units and idempotents in $\mathbb{Z}_3 \times \mathbb{Z}_6$. [3]

10. (a) Show that $4x^2 + 6x + 3$ is a unit in $\mathbb{Z}_8[x]$. [3]

(b) Show that every finite integral domain is a field. [4]

(c) Show that the ring $C[0,1]$ is not an integral domain. Characterize all units in $C[0,1]$. [3]

11. (a) Characterize those integral domains for which 1 is the only self inverse element. [3]

(b) Give examples of rings R and [3]

(i) two zero divisors $a, b \in R$ such that $a + b$ is a unit;

(ii) two units $a, b \in R$ such that $a + b$ is a zero divisor;

(iii) two idempotents $a, b \in R$ such that $a + b$ is not an idempotent.

(c) Let m and n be two positive integers and $d = \gcd(m, n)$. Show that $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$. [4]

12. (a) Let R be a finite commutative ring without zero divisors. Show that the characteristic of R is a prime number. [4]

(b) Let F be a field of characteristic p. Show that $K = \{x \in F \mid x^p = x\}$ is a subfield of F. [3]

(c) Find all solutions of the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$. Explain the reason for existence of more than two solutions in $\mathbb{Z}_{12}$. [3]

Undergraduate Examination-2025

Semester-II

Mathematics

Core Course: MJMA04

(Differential Equations-I)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer Question No. 1 and any *ten* from the rest.

1. Answer any *ten* questions from the following. [10 × 2 = 20]

   (a) Find the orthogonal trajectories of the family of circles $x^2 + y^2 = c^2$. [2]

   (b) Find the non-zero value of n for which the differential equation $\left(\frac{ny}{x^3} + \frac{y}{x^2}\right) dx + \left(\frac{1}{x^2} - \frac{1}{x}\right) dy = 0$ is an exact differential equation. [2]

   (c) Solve the differential equation $(x^3 + y^3)dx + 3xy^2dy = 0$. [2]

   (d) Solve the nonlinear differential equation $(\frac{dy}{dx})^2 + \frac{dy}{dx} - 12 = 0$. [2]

   (e) Solve: $(D+1)(D+2)(D^2+4)^4(D^2+1)^2y = 0$. [2]

   (f) Solve the differential equation $\frac{d^4y}{dx^4} = x^2 + e^{2x}$. [2]

   (g) Find a second order linear homogeneous differential equation whose linearly independent solutions are x and x^2. [2]

   (h) Solve the linear equation $x\frac{dy}{dx} - 4y = x^5e^x$. [2]

   (i) Solve the boundary value problem $\frac{d^2y}{dx^2} + 4y = 0, y(0) = 0, y(\frac{\pi}{8}) = 1$. [2]

   (j) Find two different solutions of the differential equation $\frac{dy}{dx} + |y| = 2$. [2]

   (k) Solve: $\frac{d^2y}{dx^2} = e^{2y}$. [2]

   (l) Show that, in an equation of the form $yf(xy)dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$. [2]

   (m) Find the Wronskian of the functions e^x, e^{2x} and e^{3x}. [2]

   (n) Solve the differential equation $\frac{d^2y}{dx^2} - (\frac{dy}{dx})^2 - 1 = 0$. [2]

   (o) Solve: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 7$. [2]

2. Solve the homogeneous equation $(x^2 - y^2)dx + 2xydy = 0$. [6]

3. If $e^{\int \phi(y)dy}$ is an integrating factor of the differential equation $M(x,y)dx + N(x,y)dy = 0$, then find the expression of $\phi(y)$. Hence solve the equation $ydx + 2xdy = 0$. [4+2]

4. Find the differential equation of the family of circles whose centers lie on the y-axis. [6]

5. Solve the differential equation $\frac{dy}{dx} + y = f(x)$ with initial condition $y(0) = 6$, where $f(x) = 5$, when $0 \leq x < 10$ and $f(x) = 1$, when $x \geq 10$. Hence find the value of $y(15)$. [5+1]

6. Find the orthogonal trajectories of the family of curves $r(1 + cos\theta) = 2c$, c being the parameter. [6]

7. Find a family of oblique trajectories that intersect the family of hyperbolas $xy = c$ at angle 45^o, c being the parameter. [6]

8. Solve the nonlinear differential equation $y + px = x^4p^2$. [6]

9. Reduce the equation $x^2p^2 + y(2x + y)p + y^2 = 0$ to Clairaut's form by the substitution $y = u$ and $xy = v$. Hence solve the equation. [5+1]

10. Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. [6]

11. Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{2x} + sin3x$. [6]

12. Solve the Cauchy-Euler equation $x^2y'' + 3xy' + y = x + \frac{1}{x}$, $x > 0$. [6]

13. Find an integrating factor and solve the differential equation $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$. [3+3]

14. Using the method of variation of parameters, solve the equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\, logx$, where x^3 and $\frac{1}{x}$ are two linearly independent solutions of the corresponding homogeneous equation. [6]

15. Suppose $n \neq 0, 1$. Show that the transformation $v = y^{1-n}$ reduces the Bernoulli equation to a linear equation in v. Hence solve the equation $\frac{dy}{dx} + y = xy^3$. [3+3]

16. Show that if f is any solution of the differential equation $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$, then the transformation $y = f + \frac{1}{v}$ reduces the above differential equation to a linear equation in v. Hence solve the equation $\frac{dy}{dx} = -8xy^2 + 4x(4x + 1)y - (8x^3 + 4x^2 - 1)$; given solution $f(x) = x$. [4+2]

Use separate answer script for each unit

Four Year Undergraduate Programme - 2025

Semester-II(NEP)
Mathematics
Paper: SECMA-02
(Analysis-II and Tensor Calculus)

Time: 3 Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Full Marks: 20)
(Analysis-II (Complex Analysis-I))

Answer ***any two*** questions.

1. (a) What do you mean by a bounded sequence? Show that a convergent sequence is bounded.
   (b) Prove, using ϵ-δ notations, that $\lim_{z\to\infty}\left(7+\frac{1}{z+i}\right)=7$.
   (c) If (ξ,η,ζ) and (ξ',η',ζ') are the stereographic projections of the points z and $\frac{1}{z}$ respectively, prove that $\xi=\xi'$, $\eta+\eta'=0$, $\zeta+\zeta'=1$. [(1+3)+3+3]

2. (a) Use ϵ-δ notations to prove that the function $f(z)=z^4$ is continuous at $z=a(\neq 0)$.
   (b) Show that a bounded infinite set has at least a limit point.
   (c) Find the stereographic projection of the circle $x^2+y^2+4x-6y+7=0$ in the complex plane $\mathbb{C}$ on the Riemann sphere $\xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2=\frac{1}{4}$. [3+4+3]

3. (a) When is a function f said to be continuous at a point z_0? Explain an example by ϵ-δ notations.
   (b) Define limit points of a sequence $\{z_n\}$. Find the limit points of $\left\{(-i)^n\left(1+\frac{3i}{n}\right)\right\}$.
   (c) Prove that a complex function continuous on a closed and bounded region is bounded. [(1+2)+3+4]

Unit-II (Full Marks: 40)
(Tensor Calculus)

Answer ***any eight*** questions.

1. Define scalar, vector, and tensor fields. Explain covariant, contravariant, and mixed tensors and their order and rank. A field $A(p,q,r,s)(\tilde{x})$ of coordinates $\tilde{x}=(x^1,x^2,x^3,\cdots,x^n)$ is such that their relations in two coordinate systems $\tilde{x}$ and $\tilde{\bar{x}}(=(\bar{x}^1,\bar{x}^2,\cdots,\bar{x}^n))$ are

$$\bar{A}(i,j,k,l)(\tilde{\bar{x}})=\frac{\partial\bar{x}^k}{\partial x^s}\frac{\partial\bar{x}^i}{\partial x^r}\frac{\partial x^q}{\partial\bar{x}^j}\frac{\partial x^p}{\partial\bar{x}^l}A(p,q,r,s)(\tilde{x}),\quad p,q,r,s,i,j,k,l=1,2,\cdots,n.$$

   Examine whether the field is a tensor field. Find its order and rank, if yes. [5]

2. Transform the tensor field $(2x+y,xyz,(x+y)^2z)$ in Cartesian coordinates to the contravariant and covariant components in spherical polar coordinates. [5]

3. Define inner multiplication, contraction, and quotient law of tensor fields. If A_i^{pqr} and B_{st}^{lm} are two tensor fields, examine which one among their sum, difference $A_i^{pqr}\pm B_{st}^{lm}$, and the product $A_i^{pqr}B_{st}^{im}$ is a tensor field. Justify the answer. Prove that the contraction of B_i^j is a scalar field. [5]

4. Define symmetric and skew-symmetric tensors. Given A^{ijk} and B_{lmn} are two skew-symmetric tensors. Verify the (symmetric/skew-symmetric) nature of $C^{ij}_{lm} = A^{ijk}B_{lmk}$. If $\Phi = a_{pq}A^pA^q$, show that Φ can be expressed as $\Phi = b_{ij}A^iA^j$, where b_{ij} is a symmetric tensor field. [5]

5. Let a quantity $A(i,j,k)$ be such that $A(i,j,k)B^{jl}_i = C^l_k$, where B^{jl}_i is an arbitrary tensor field and C^l_k is a mixed tensor of rank $(1,1)$. Prove that $A(i,j,k)$ is a tensor field. Determine its rank. [5]

6. Prove that the partial derivative of a scalar field is a vector field, while the partial derivative of a vector or a tensor field is not a tensor field. Define covariant derivative. Write down the formula for the covariant derivative of the tensor field $A^{ij}_{klm}(\tilde{x})$ with respect to the variable x^r. [5]

7. Define the line element of a curve in space. Show that the square of the line element can be expressed as a quadratic form. What are the names of the coefficients involved in the quadratic form? Do they behave like a tensor? Examine whether it is symmetric or anti-symmetric, if so. Provide a geometric interpretation of the coefficients. [5]

8. Find metric tensors g_{ij} and its associated tensors g^{ij} in the cylindrical polar coordinates (ρ, ϕ, z) and spherical polar coordinates (r, θ, ϕ). [5]

9. Define Christoffel's symbols of the first and second kind. Evaluate the expressions for Christoffel's first- and second-kind symbols in orthogonal curvilinear coordinates in terms of partial derivatives of metric tensors. Hence, obtain their expressions for Cartesian coordinate systems. [5]

10. Derive the equation for a geodesic on a curved surface. Hence, find two geodesics on a cylindrical surface of radius a, passing through a given point on it. Interpret those geodesics. [5]

11. Write the formula for the second-order covariant derivatives of a vector field $B_i(\tilde{x})$ with respect to two variables x^l, x^m (say). Is it commutative under the interchange of derivatives with respect to x^l, x^m? If no, find the difference $(B_{i,lm} - B_{i,ml})$, and examine whether this difference can be written as a sum of $B_l(\tilde{x})$. Hence or otherwise define the Riemann-Christoffel tensor. [5]

12. Prove the formulae a) $divA^p = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^k\right)$, b) $\Gamma^i_{ij} = \frac{\partial}{\partial x^j}\sqrt{g}$. [5]