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each unit

Undergraduate Examination-2023

Semester-I
Mathematics
Course: MJMA 01
(Algebra I and Analysis I)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Algebra I (Full Marks: 40)

Answer *any four* questions.

1. (a) Let G be a set with n elements. How many binary operations can you define on G?
(b) Let $G = \{-1, 1, i, -i\}$ where $i = \sqrt{-1}$. Prove that G is a cyclic group under multiplication.
(c) Is the group $GL_2(\mathbb{R})$ of invertible 2×2 real matrices cyclic? Justify your answer. [2+5+3]

2. (a) Prove that the order of every element of a finite group is finite.
(b) Let x, y and z be elements of a group G such that $xyz = e$. Does it follow that $yzx = e$?
(c) Let G be a group such that $g^2 = e$ for all $g \in G$. Prove that G is abelian. [3+3+4]

3. (a) Prove that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication for any integer $n \geq 2$.
(b) Let G be an infinite group and let $H_1, H_2, \ldots$ be a collection of groups such that $H_n \subseteq H_{n+1}$ for all $n \geq 1$. Prove that $\bigcup_{n=1}^{\infty} H_n$ is a subgroup of G.
(c) Let G be a finite group of order $n > 2$. Prove that G does not have any subgroup of order $n - 1$. [3+3+4]

4. (a) Define the center $Z(G)$ of a group G. Determine, with proof, $Z(Q_8)$ and $Z(S_n)$ for $n \geq 3$.
(b) What is the maximum possible order of an element of S_7? Justify your answer.
(c) For $n \geq 4$, prove that the number of elements in S_n which are the product of two disjoint 2-cycles is $\frac{n(n-1)(n-2)(n-3)}{8}$. [5+2+3]

5. (a) Write the elements of the group $(\mathbb{Z}/18\mathbb{Z})^*$ and justify whether it is a cyclic group or not.
(b) Let G be a group of order 28. Prove that G has a non-trivial proper subgroup.
(c) Prove that every non-abelian group has a non-trivial cyclic subgroup. [4+3+3]

6. (a) Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.
(b) Determine, with proof, the smallest subgroup of $(\mathbb{Z}, +)$ containing $6, 8$ and 10.
(c) Let H be a subgroup of $(\mathbb{Q}, +)$ with the property that $\frac{1}{x} \in H$ for every non-zero element $x \in H$. Prove that either $H = \{0\}$ or $H = \mathbb{Q}$. [4+3+3]

Unit-II : Analysis I (Full Marks: 40)

Answer **any four** questions

1. (a) Prove the following results:
   (i) If $a \neq 0$ and $b \in \mathbb{R}$ such that $a.b = 1$ then $b = \frac{1}{a}$.
   (ii) If $a.b = 0$ then either $a = 0$ or $b = 0$. [5]

   (b) For any three elements $a, b, c \in \mathbb{R}$ prove the following results :
   (i) If $a > b$ and $b > c$ then $a > c$.
   (ii) If $a > b$ and $c > 0$ then $ca > cb$. [5]

2. (a) Let $x, y \in \mathbb{R}$ and $x < y$. Then prove that
   (i) there is a rational number r such that $x < r < y$. [5]
   (ii) there is an irrational number s such that $x < s < y$. [5]

   (b) Show that $\mathbb{Q}$ (set of all rational numbers) is countable.

3. (a) Define Cauchy sequence and give an example for the same. Prove that every Cauchy sequence is bounded. Does the converse hold? Justify. [5]

   (b) Examine if the sequence $x_n = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)^{n+1}}{n}$ is a Cauchy sequence. [5]

4. (a) Prove that every monotone decreasing sequence which is bounded below is convergent and converges to its greatest lower bound. [5]

   (b) Evaluate $\lim\limits_{n\to\infty} \frac{1}{n} \{(a+1)(a+2)\cdots(a+n)\}^{\frac{1}{n}}$ if $a > 0$. [5]

5. (a) Define subsequence of a sequence. Prove that if a sequence is convergent then every subsequence of it is convergent. Does the converse hold? Justify. [4]

   (b) State and prove nested intervals theorem. [6]

6. (a) Give $(\epsilon - \delta)$ definition of the limit of a function and hence evaluate $\lim\limits_{x\to 3} f(x)$ where $f(x) = x^3 + 3x^2 + 3x + 5$, $x \in [2, 5]$. [5]

   (b) If $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$ then prove that $\lim\limits_{x\to a} f(x)g(x) = lm$. [5]

Use separate answer script for each unit

Undergraduate Examination-2023

Semester-I
Mathematics
Course: MJMA 02
(Analytical Geometry and Vector Calcus)

Time: Three Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Unit I: Analytical Geometry (Full Marks: 40)

Answer ***any four*** questions.

1. (a) Find the angle by which the axes should be rotated so that the equation $ax^2 + 2hxy + by^2 = 0$ becomes another equation in which the term xy is absent. Hence find the angle through which the axes are to be rotated so that the equation $17x^2 + 18xy - 7y^2 = 1$ may be reduced to the form $Ax^2 + By^2 = 1,\ A > 0$. [5]

   (b) Transform the equation $x^2 - 5xy + y^2 + 8x - 20y + 15 = 0$ to its canonical form and determine the nature of the conic. [5]

2. (a) Prove that the locus of the mid-points of the family of focal chords of the conic $\frac{l}{r} = 1 + e\cos\theta$ is $r(1 - e^2\cos^2\theta) + el\cos\theta = 0$. [5]

   (b) Find the equation of the common tangent to the circle $x^2 + y^2 = 4ax$ and the parabola $y^2 = 4ax$. [5]

3. (a) A plane passes through a fixed point (p, q, r) and cuts the co-ordinate axes at A,B and C. Show that the locus of the centre of the sphere OABC is $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$. [5]

   (b) Find the centre and the radius of the circle

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0,\ x + 2y + 2z = 15.$$

   [5]

4. (a) Prove that a homogenous second degree equation in x, y, z represents a cone whose vertex is at origin. [5]

   (b) Find the equation of the straight lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Find also the angle between them. [5]

5. (a) Prove that the plane $ax + by + cz = 0\ (a, b, c \neq 0)$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. [5]

   (b) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. [5]

6. (a) Find the equation of the tangent plane to the paraboloid $ax^2 + by^2 = 2z$ which is parallel to the plane $lx + my + nz = 0$. [5]

   (b) Find the equation of the generators of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$. [5]

Unit-II : Vector Calculus (Full Marks: 40)

Answer ***any four*** questions

1. (a) Using vector method, prove that the diagonals of a parallelogram bisect each other.

   (b) Prove, using vector method, that the medians of a triangle are concurrent.

   (c) Let $\vec{A}, \vec{B}$ and $\vec{C}$ be three vectors on $\mathbb{R}^2$. Prove that $|\vec{A}+\vec{B}+\vec{C}| \leq |\vec{A}|+|\vec{B}|+|\vec{C}|$. [4+4+2]

2. (a) Determine the equation of the plane perpendicular to the vector $2\hat{i}-3\hat{j}+6\hat{k}$ and pssing through the terminal point of the vector $\hat{i}+2\hat{j}+3\hat{k}$.

   (b) Determine the acute angle formed by the two main diagonals of a cube.

   (c) Let $\vec{A}, \vec{B}$ and $\vec{C}$ be three vectors in $\mathbb{R}^3$. Prove that $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=\vec{0}$. [3+3+4]

3. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function. Prove, using vector method, that the length of the curve between the points $(a, f(a))$ and $(b, f(b))$ is $\int_a^b \sqrt{1+(f'(x))^2}dx$.

   (b) For the differentiable curve $\vec{R}(t) = 3\cos t\ \hat{i} + 3\sin t\ \hat{j} + 4t\ \hat{k}$, find the unit tangent vector $\vec{T}$, principal normal $\vec{N}$, the binormal vector $\vec{B}$, curvature κ, torsion τ and verify the Frenet-Serret formula. [4+6]

4. (a) Prove that if $f:\mathbb{R}^3\to\mathbb{R}$ is defined by $f(x,y,z)=x^2\sin y+y^2\cos z$, then $\nabla\times(\nabla f)=\vec{0}$.

   (b) Can the vector field $F(x,y,z)=x(y^2+1)\hat{i}+(ye^x-e^z)\hat{j}+x^2e^z\hat{k}$ be the curl of some vector field $G:\mathbb{R}^3\to\mathbb{R}^3$? Justify your answer.

   (c) Let $\vec{r}(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}$ and $r(x,y,z)=\sqrt{x^2+y^2+z^2}$. For any integer $n\geq 1$, prove that $\nabla\cdot(r^n\vec{r})=(n+3)r^n$. [3+3+4]

5. (a) Determine real numbers a, b and c such that the vector field $F(x,y,z)=(-4x-3y+az)\hat{i}+(bx+3y+5z)\hat{j}+(4x+cy+3z)\hat{k}$ is irrotational.

   (b) Determine the flow lines of the constant vector field $F(x,y,z)=2\hat{i}-3\hat{j}+\hat{k}$.

   (c) Let $f:\mathbb{R}^3\to\mathbb{R}$ be of class $\mathcal{C}^2$. Prove that the curl of grad(f) is $\vec{0}$. [3+3+4]

6. (a) Determine the point on the plane $2x-3y-z=4$ that is closest to the origin.

   (b) Determine three positive real numbers whose sum is 18 and whose product is as large as possible.

   (c) Determine the critical points of the function $f(x,y)=e^{-(x^2+y^2)}$ and find the nature of the critical points using Hessian criterion. [4+3+3]

Undergraduate Examination-2023

Semester-I

Mathematics

Course : SECMA 01 (Algebra II and Analysis II)

Time: Three Hours **Full Marks: 60**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.
(*Use separate answer book for each unit*)

UNIT - I : Algebra II (Marks : 40)

Answer ***any four*** questions.

1. (a) Define a consistent system of linear equations. What is the nature of the solution(s) of a consistent linear system of three equations in four variables. [1+2]

   (b) Let the linear system

   $$x + 3y = f$$
   $$cx + dy = g$$

   be consistent for every $(f, g) \in \mathbb{R}^2$. What can you say about the coefficients c and d? If $f = g = 0$, find the nature of the solutions. [2+1]

   (c) Suppose that for the linear system $AX = b$, the coefficient matrix A has a pivot position in every row. Show that the system is consistent? What happens if A has a pivot position in every column? [2+2]

2. (a) Is $b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}$? Justify. [3]

   (b) Find the value of h so that $b \in span\{a_1, a_2\}$, where $a_1 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$, $a_2 = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 1 \\ h \end{pmatrix}$. [2]

   (c) Find the $span\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\}$. [2]

   (d) Find a condition so that $b = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$ is in $span\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\}$. [3]

3. (a) Let A be an $m \times n$ matrix. Show that the columns of A spans $\mathbb{R}^n$ if and only if for each $b \in \mathbb{R}^n$, the equation $AX = b$ has a solution. [4]

   (b) Let $A = \begin{pmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -2 & 1 & -4 \\ 2 & 0 & 3 & 1 \end{pmatrix}$, Does the equation $AX = b$ have a solution for each $b \in \mathbb{R}^4$? [3]

   (c) Let A is a 3×3 matrix. If there is a vector $b \in \mathbb{R}^3$ such that the equation $AX = b$ has a unique solution, then show that the columns of A spans $\mathbb{R}^3$. [3]

4. (a) Let A be a 7×5 matrix. If the columns of A are linearly independent, find the number of pivot columns of A. Does $AX = 0$ have unique solution? [3]

   (b) Let A be an $m \times n$ matrix. Show that the columns of A are linearly independent if and only if for each $b \in \mathbb{R}^m$, the equation $AX = b$ has at most one solution. [4]

   (c) Show that three vectors $a_1, a_2, a_3 \in \mathbb{R}^3$ are linearly independent if and only if they can not be on the same plane. [3]

5. (a) Define subspace and a basis of a subspace. Show that the null space of a matrix is a subspace. [3]

   (b) Let U be a subspace of $\mathbb{R}^n$. If U has a basis of k elements, then show that a linearly independent subset of U can have at most k vectors. [4]

   (c) Find a basis and the dimension of the subspace $V = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + 3z = 0\}$. [3]

6. (a) If possible, construct a 3×4 matrix A such that the $\dim N(A) = 2$ and $\dim C(A) = 2$. Otherwise, explain why it is not possible. [3]

(b) Let U be the reduced row echelon form of a matrix A. Show that the nonzero rows of U form a basis of the row space $R(A)$ of A. [4]

(c) Let A be a 3×3 matrix whose row space is the $span\{(3, 0, -2), (1, 2, 0)\}$. Find the null space of A. [3]

UNIT - II : Analysis II (Marks : 20)

Answer ***any two*** questions.

1. (a) Find y_n where
(i) $y = e^x \cos x$. (ii) $y = \tan^{-1}(\frac{x}{a})$. [4]

(b) If $y = \sin(m \sin^{-1} x)$, prove that
(i) $(1 - x^2)y_2 = xy_1 - m^2 y$,
(ii) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$. [2+4]

2. (a) State and prove Lagrange's Mean Value theorem. [6]

(b) In the Mean Value Theorem $f(x + h) = f(x) + hf'(x + \theta h)$, $0 < \theta < 1$, show that the limiting value of θ as $h \to 0^+$ is $\frac{1}{2}$ where $f(x) = \sin x$. [4]

3. (a) Evaluate: $\lim_{x \to 0} (\cos x)^{\cot^2 x}$. [4]

(b) If $\lim_{x \to 0} \frac{a \sin x + \sin 2x}{x^3}$ is finite then find the value of a and the limit. [6]